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Mascheroni constructions

JULIUS H. HLAVATY, *Dewitt Clinton High School, New York City.*

A good, understandable approach to Euclidean constructions performed with compasses alone.

Introduction

Plato is credited with (or blamed for) restricting the geometer to the use of the compasses and straightedge alone. As Hogben points out, it was perfectly consistent for Plato to lay down this limitation: "Geometry was an aid to spiritual perfection. We are not expected to attain spiritual perfection and enjoy ourselves at the same time. So it was natural for those who held this belief to make geometry as difficult and unpalatable as generations of school children have found it." However that may be, we must not overlook that it was this very restriction which opened the way to a great deal of investigation in mathematics arising from attempts to solve certain problems by means of the straightedge and compasses alone. The long history of the three notorious problems of antiquity—the trisection of an angle, the doubling of the cube, and the squaring of the circle—owes its start to the Platonic restriction. The high point of the history of construction problems involving the straightedge and compasses was reached when, in the last century, it was demonstrated that all constructible numbers are algebraic, and when the impossibility of the solution of the three famous problems was proved.

It is natural to expect—and the expectation has been realized by experience—that if we permit the use of instruments other than the straightedge and the compasses, we can solve a greater variety of problems. For example, it is well known that the trisection of an angle is possible if we permit just a slight modification of the straightedge, e.g., putting one mark on the straightedge.

The Mascheroni problem

It was left for an Italian mathematician (and it turned out later that he had been anticipated by a Danish mathematician) to prove that Plato missed a trick. In 1797, Luigi Mascheroni published his book *Geometria del Compasso* in which he showed that *all* the constructions of geometry that are performable by the use of the straightedge and compasses can also be performed by means of the compasses *alone*. The Danish mathematician Hjelmslev discovered in 1928 that one of his countrymen, Georg Mohr, had anticipated Mascheroni in his discovery. In *Euclides Danicus*, published in 1672, Mohr had solved the so-called Mascheroni problem.

The question naturally arises: "How can it be shown that all the constructions of Euclidean geometry can be done by means of the compasses alone?"

The number of constructions in geometry is infinite, and therefore we certainly cannot prove our problem by solving every construction problem with compasses alone. However, all Euclidean constructions are merely finite successions of the following four fundamental constructions:

I Drawing a circle with a given center and a given radius

II Finding the points of intersection of two circles

III Finding the points of intersection of a line and a circle

IV Finding the point of intersection of two lines

It is obvious that the first two problems offer no difficulty since our instruments are the compasses. It is necessary, however, to make some convention about the meaning of a straight line, since a straight line cannot be drawn with compasses alone. We will say that a straight line is given if two of its points are given. It must be possible, however, to find as many points on the line so given as we desire. The solution of this problem is easy and will be made clear in the rest of the article.

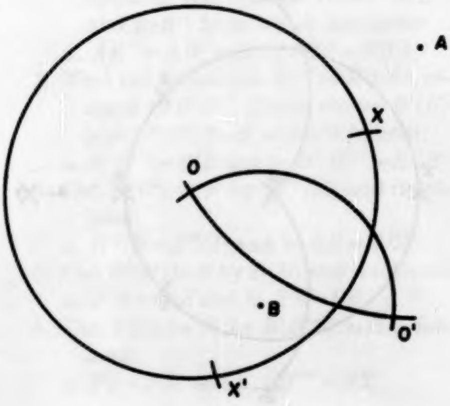
We must therefore show that it is possible to solve problems III and IV with the compasses alone. The solution of these two problems, as well as many other Mascheronian constructions, can be simplified considerably by introducing the theory of inversions. However, since that theory is not a usual part of elementary geometry, we will give the constructions and solutions that do not explicitly involve this theory. The reader may consult Courant or Yates (see the bibliography) for the constructions by use of inversions. We will conclude with additional problems—some worked out and others merely proposed.

Solution of problem III

Problem: To find the points of intersection of a line and a circle

Case 1: Given circle O with radius r and a line defined by points A and B, where A and B are not collinear with O

Figure 1



Construction:

1. With A as center and radius AO draw an arc through O (problem I).
2. With B as center and radius BO draw an arc through O (problem I).
3. Determine O' point of intersection of these two circles (problem II).
4. With O' as center and radius r intersect circle O in X and X' (problems I and II).

Points X and X' are the required points of intersection.

Proof:

$$AO = AO', \quad BO = BO'$$

by construction.

AB is the perpendicular bisector of OO'. (Two points equally distant from the ends of a line segment determine . . .)

$$OX = O'X, \quad OX' = O'X'$$

by construction.

X and X' lie on AB.
(Points equally distant from the ends of a line lie on the perpendicular bisector.)

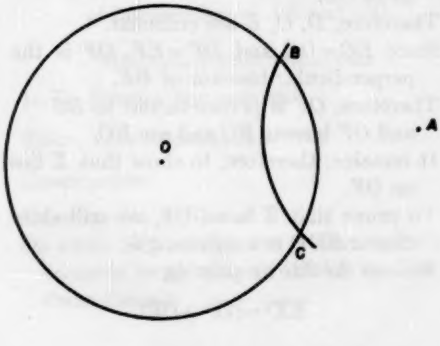
Case 2: Given circle O with radius r, and a line through the center of the circle and determined by A and O

We can draw the arc of an arbitrary circle with center A and intersecting circle O in points B and C (Problems I and II).

It is clear that the midpoints of the arcs BC are the required points.

Therefore, if we succeed in bisecting an arc we have solved the problem.

Figure 2



Auxiliary problem IIIa: To bisect an arc of a circle

Given: Arc BC with center O and radius r

To find: X, the midpoint of arc BC

Construction:

With B and C as centers and radius r draw arcs DO and OE.
Construct $OD=OE=BC$.
Determine point F such that

$$DF=EF=CD(=BE).$$

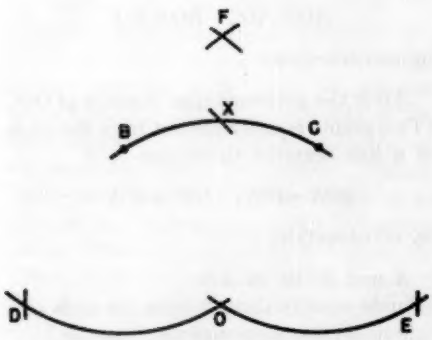


Figure 3

With OF as radius and D (or E) as center cut arc BC in X.

Point X is the required midpoint of arc BC.

Proof:

Since $BC=DO$ and $BD=OC$, BC is parallel to DO.
Since $BC=OE$ and $BO=CE$, BC is parallel to OE.
Therefore, D, O, E are collinear.
Since $DO=OE$ and $DF=EF$, OF is the perpendicular bisector of DE.
Therefore, OF is perpendicular to BC and OF bisects BC and arc BC.
It remains, therefore, to show that X lies on OF.
To prove that X is on OF, we will show that $\angle XOE$ is a right angle.
We can do this by proving

$$\overline{EX}^2=\overline{OX}^2+\overline{OE}^2.$$

Since CO is the median of triangle CDE,

$$4\overline{OC}^2=2\overline{DC}^2+2\overline{CE}^2-\overline{DE}^2.$$

But by construction $DE=2OE$, $DC=EF$, $OC=CE$.

$$4\overline{OC}^2=2\overline{EF}^2+2\overline{OC}^2-4\overline{OE}^2$$
$$4\overline{OE}^2+2\overline{OC}^2=2\overline{EF}^2$$
$$2\overline{OE}^2+\overline{OC}^2=\overline{EF}^2$$

In right triangle OEF, $\overline{EF}^2=\overline{OE}^2+\overline{OF}^2$

$$2\overline{OE}^2+\overline{OC}^2=\overline{OE}^2+\overline{OF}^2$$
$$\overline{OE}^2+\overline{OC}^2=\overline{OF}^2.$$

Since X is on arc BC, $OC=OX$; and $OF=EX$ by construction,

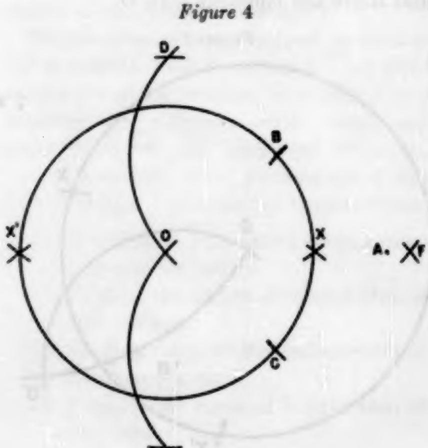
$$\overline{OE}^2+\overline{OX}^2=\overline{EX}^2.$$

We have now demonstrated that it is possible to bisect an arc, and, thus, we can complete the solution of Problem III.

The complete construction may then be carried out as follows:

With center A and any radius (we have used the radius of the given circle, for convenience) intersect circle O in points B and C.
With B and C as centers and radius BO draw arcs OD and OE.
With BC as radius and O as center intersect arcs OD and OE in D and E.
With centers D and E and radius DC construct arcs intersecting in F.

Figure 4



With center E and radius OF intersect circle O in points X and X'.

We can check our construction by cutting the circle O with a circle with center D and radius OF.

Solution of problem IV

Problem: To find the point of intersection of two lines AB and CD

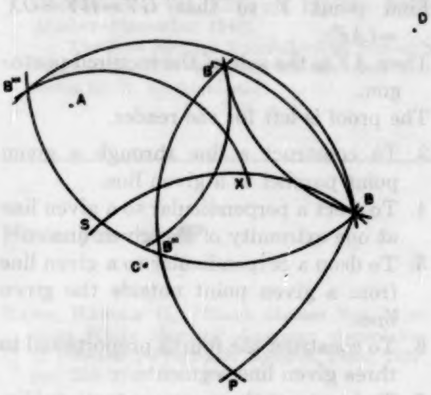


Figure 5

Construction:

Let us adopt the following notation: $K(Y)$ is to mean: "A circle with K as center and passing through Y and having therefore the radius KY."

1. Find the symmetric B' of B with respect to CD. (Draw circles $C(B)$ and $D(B)$ from which it follows:
   a. $DB'=DB$ and b. $CB'=CB$)
2. Find the symmetric B'' of B' with respect to AB. (Draw circles $A(B')$ and $B(B')$ from which it follows:
   a. $AB''=AB'$ and b. $BB''=BB'$)
3. Find the symmetric B''' of B with respect to $B'B''$. (Draw circles $B'(B)$ and $B''(B)$ from which it follows:
   a. $B'B'''=B'B$ and b. $B''B'''=B''B$)
4. Cut $B(B')$ in P by $B'''(B)$ and it follows:
   a. $B'''B=B'''P$ and b. $BP=BB'$
5. Cut $B'(B)$ in S by $P(B)$ and it follows:
   a. $B'B=B'S$ and b. $PS=PB$
6. Cut $P(B)$ in X by $S(B''')$ and it follows:
   a. $PB=PX$ and b. $SB'''=SX$

X is the required point of intersection of AB and CD.

Proof:

It has to be proved that X lies on AB and on CD.

1. $B'(S)=P(S)$
   (Since $B'S=BB'$ [5a], $BB'=PB$ [4b], $PB=PS$ [5b])
2. $SX=SB'''$ by 6b
3. arc SX = arc SB'''
4. $\angle SBX=\frac{1}{2}$[arc SX]
   $\angle SBB'''=\frac{1}{2}$[arc SB''']
5. and $\angle SBX=\angle SBB'''$
   and X lies on BB'''
6. AB is the perpendicular bisector of $B'B''$ by 2 above.
7. $B''B'''=B'''B'$
   (Since $B''B'''=B''B$ [3b], $B'''B'=B'B$ [3a], $B''B=B'B$ [2b])
8. Therefore, X lies on AB.
9. Triangles $B'''BP$ and XBP are isosceles and have $\angle XBP$ in common. ($B'''B=B'''P$ [4a], $PX=PB$ [6a])
10. Triangles $B'''BP$ and XBP are similar.
11. $BB''':PB=PB:BX$
12. $BB''':BB'=BB':BX$,
    by substitution in 4b.
13. $\angle B'''BB'=\angle XBB'$,
    by identity.
14. Triangles $B'''BB'$ and BXB' are similar,
15. But triangle $B'''BB'$ is isosceles, by 3a.
16. Therefore, triangle BXB' is isosceles and $BX=B'X$.
17. And since CD is the perpendicular bisector of BB' (by 1),
18. X must lie on CD, and X is the point of intersection of AB and CD.

Miscellaneous problems

1. To bisect a line segment

Given: Line segment AB

Construction:

Draw circles $A(B)$ and $B(A)$.

On circle $B(A)$ find point C diametrically opposite to A. (Mark off AB on $B(A)$ three times.)

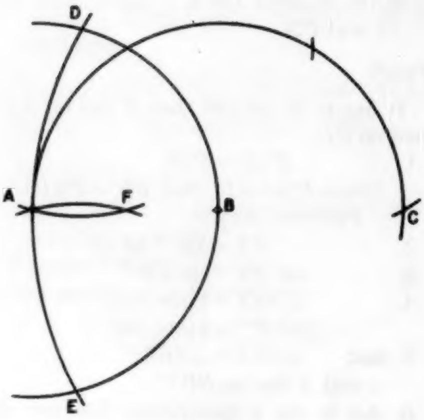


Figure 6

Draw arc with $C(A)$ intersecting $A(B)$ in D and E.

Draw $D(A)$ and $E(A)$ intersecting in F. F is the required point.

Proof:

Since $AD=AE$, $CD=CE$ and $DF=EF$, AC is the perpendicular bisector of DE and F lies on CA.

Isosceles triangles ADF and CAD are similar.

Therefore

$$CA:AD=AD:AF$$

or

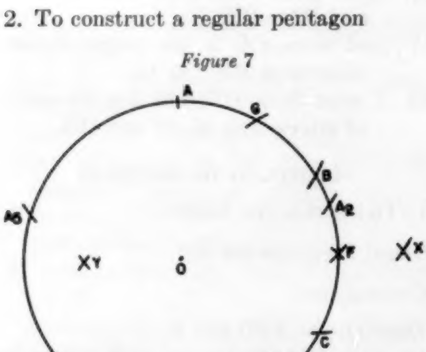
$$2AB:AB=AB:AF$$

and

$$AB=2AF.$$

2. To construct a regular pentagon

Figure 7



Construction:

Take any point A on circle O.

Find point D diametrically opposite to A.

Find F, the midpoint of arc BC. (See auxiliary construction in the solution of Problem III above.)

From F mark off points G, H so that $FG=FH=AO$.

Find point Y so that $GY=HY=OX$ $=(AF)$.

Then AY is the side of the required pentagon.

The proof is left for the reader.

3. To construct a line through a given point parallel to a given line.
4. To erect a perpendicular to a given line at one extremity of the given line.
5. To drop a perpendicular to a given line from a given point outside the given line.
6. To construct the fourth proportional to three given line segments.
7. To construct the mean proportional between two given line segments.
8. To construct the tangents to a given circle from a given external point.
9. To find the center of a given circle. (This is called Napoleon's problem. Napoleon is reported to have challenged Mascheroni to solve this problem, and Mascheroni succeeded with a very elegant solution.)

Bibliography

Courant, Richard and Robbins, Herbert. *What Is Mathematics?* New York: Oxford University Press, 1941.

Among the many excellent things in this book is a good and complete treatment of geometrical constructions. The theory of inversions is introduced in the treatment of the Mascheroni problem.

There is also a treatment of the other, analogous problem of construction—restriction to the use of the straightedge alone. Steiner's solution of these constructions is presented, using the straightedge and a given circle with a given center.

Constructions with other tools, mechanical instruments, and linkages are also dealt with.

Lanascol, A. Quemper de. *Géométrie du Compas.* Paris, 1925.

The fifteen chapters of this book are a treasure-trove of constructions. All the con-

structions of elementary geometry are carefully carried out and demonstrated. For many of the problems there are many alternate solutions given, and there is a wealth of historical information on various solutions. For example, there are four distinct solutions of the construction of the fourth proportional. Mascheroni's own constructions are given for many of the problems.

STEINER, JACOB. "Geometrical constructions with a ruler, given a fixed circle with its center." Translated by M. E. Stark. *Scripta Mathematica*. Vol. XIV, pp. 187–264. (September–December 1948)

The first English translation of Steiner's original booklet with an introduction and notes by R. C. Archibald.

YATES, ROBERT C. *Geometrical Tools, A Mathematical Sketch and Model Book*. St. Louis: Educational Publishers, revised 1949.

This is an excellent little book, published in workbook form, with constructions to be done in the book. Our special problem is treated on pages 42–53, and the approach is through inversions. The topics covered in this book are: the straightedge and modern compasses; dissection of plane figures; the compasses; folds and creases; the straightedge; line motion linkages; the straightedge with immovable figure; the assisted straightedge; parallel and angle rulers; higher tools and quartic systems; general plane linkages. Complete bibliographies are given on each of these topics.

Have you read?

HAND, HAROLD C., "Black Horses Eat More than White Horses," *American Association of University Professors Bulletin*, June, 1957, pp. 266–279.

Are today's high school students being denied the opportunity to study mathematics and science that those of 1900 had? With today's almost feverish push for technological training, we must be careful to interpret properly the data bearing on such questions.

The author of this article analyzes statistics from the U. S. Office of Education to answer this question and related ones in the negative. It is interesting to note that this article is directed toward clarifying a misinterpretation of the same statistics used to prove that "We Are Less Educated Than 50 Years Ago," an article headlined on the cover page of the November 30, 1956, issue of *U. S. News & World Report*.—LYMAN C. PECK, *Ohio Wesleyan University, Delaware, Ohio.*

JENSEN, ARTHUR, "Computation With Approximate Numbers," *California Journal of Secondary Education*, January, 1957, pp. 14–15.

You should read this article because it points out a blind spot in most of our teaching. The author compares the failure to teach computation with approximate numbers to the teaching that Paul Revere and George Washington were brothers. He feels teaching students to find the circumference of a 3" circle by multiplying by 3.1416 is just as gross an error. However, you will be surprised that he does not blame the mathematics teachers but the administrators of the school, who apparently do not realize such atrocities are being committed in the school. In any event you must agree that this field of approximate measure and its use is sadly neglected.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Mathematics of Finance, Hugh E. Stelson. New York, D. Van Nostrand Company, Inc., 1957. Cloth, xii+327 pp., $5.50.

This book is designed for use in a college course taken by students who are interested in obtaining a major in mathematics or business administration. While a review of the algebra needed for the course is included in the book, the student should have had a course in college algebra before studying *Mathematics of Finance*. In order to cover the topics thoroughly, a two-semester course would be required.

One of the outstanding features of this book is the attempt to bring into the work many problems which are likely to be met by the students in life situations. The author also tries to define and explain many of the terms and procedures used in the business world. At times he becomes so engrossed in doing this that the explanations of the mathematical processes to be used are neglected. It is in these cases that it seems as if the book is written primarily for business administration students.

Many of the explanations of specific topics are very brief and would need to be supplemented extensively by the teacher. The illustrative problems included are for the most part quite good. In my opinion more exercises should have been included. Since in most cases there are only a few following the explanation of a particular topic, the teacher is not given much opportunity for selection. A good feature is a list of problems covering the entire chapter included at the end of many chapters.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The foundations of algebra

CHARLES BRUMFIEL, *Ball State Teachers College, Muncie, Indiana.*

Some excellent ideas anent introducing proofs in high school algebra.

IN THE YEARS immediately ahead we shall undoubtedly see much experimentation in the teaching of algebra. The trend will be toward a presentation that stresses the importance of assumptions, definitions, proofs, and the logical concepts utilized in proofs. How quickly can we move in this direction? Teachers will be encouraged to experiment if suitable materials are placed in their hands.

This article is not to be considered a teachable unit in beginning algebra. But it illustrates the kind of reasoning we should like to obtain from algebra students. In a teaching situation many numerical examples should precede the theorems. Someone has remarked pessimistically that the per cent of young people in our schools who can follow careful deductive reasoning is vanishingly small. However, we always have a few who can do so. It is hoped that some teachers of mathematics will be able to use the ideas in this article to construct experimental algebra units. We need to find out how far we can go in our presentation of algebra as a logical structure rather than a hodgepodge of rules and descriptions.

As a basis for the construction of the complex number system studied in high school algebra, let us assume the existence of a set

$$S=\{0, 1, 2, 3, 4, \cdots\}$$

that consists of zero and the natural numbers. Also let us agree that operations called addition and multiplication are defined so that every ordered pair of numbers in S has a unique sum and product. In particular, sums and products of natural numbers are natural numbers. We now postulate that these operations have the following properties:

For all numbers a, b, c in S it is true that:

A 1. $a+b=b+a$	M 1. $ab=ba$
A 2. $(a+b)+c=a+(b+c)$	M 2. $(ab)c=a(bc)$
	M 3. $a\cdot 1=a$
A 3. $a+0=a$	M 4. $ab=ac$ and $a\neq 0\rightarrow b=c$
A 4. $a+b=a+c\rightarrow b=c$	

D. $a(b+c)=ab+ac$

The above postulates can be used to illustrate the processes of deductive reasoning. If the teacher is skillful, the development of algebra from these postulates should not involve logical difficulties greater than those encountered in geometry. We present below a collection of theorems, definitions, and problems that might be useful to the algebra teacher who would like to follow the advice of modern algebraists and stress the importance of definition, proof, and logical structure in algebra.

We sketch proofs of a few theorems that are a consequence of the nine basic postulates above. All variables range over S, that is, letters are used to represent numbers in S alone. One remark is in order. The sign "$=$" will only be used to indicate *identity*. The statement "$a=b$" means that "a" and "b" are symbols for the same number in S. Hence, it is needless to remark that $x=y=z\rightarrow x=z$.

Theorem 1: For every x in S, $1\cdot x=x$.

Proof: $1\cdot x=x\cdot 1$ M 1

$x\cdot 1=x.$ M 3

The reader may wonder why anything as obvious as Theorem 1 needs proof. Even the simple statement, $1\cdot 2=2$, does not occur among our postulates. However, we can find the two statements, $1\cdot 2=2\cdot 1$ and $2\cdot 1=2$. Our proof consists of calling attention to the fact that we have made these two assumptions, and together they imply that $1\cdot 2=2$.

Theorem 2: For every a in S, $a\cdot 0=0$.

Proof:

$a(1+0)=a\cdot 1+a\cdot 0=a+a\cdot 0$ — D and M 3

$a(1+0)=a\cdot 1=a+0$ — A 3 and M 3

$a+0=a+a\cdot 0\rightarrow 0=a\cdot 0$ — A 4

Theorem 3: $(a+b)+(c+d)=(a+(c+d))+b$.

Proof:

$(a+b)+(c+d)=a+(b+(c+d))$ — A 2

$a+(b+(c+d))=a+((c+d)+b)$ — A 1

$a+((c+d)+b)=(a+(c+d))+b.$ — A 2

Many theorems like Theorem 3 can be proved using the commutative laws A 1, M 1 and the associative laws A 2, M 2. Indeed, a proof employing mathematical induction enables one to establish general associative and commutative laws, showing that if arbitrarily many numbers of S are to be added they can be reordered in any desired manner (commutativity) and then grouped by parentheses in any fashion (associativity). It follows that the order of performing a series of additions need not be indicated by parentheses, and we may write without ambiguity "$a+b+\cdots+l$," since any insertion of parentheses leads to the same sum. We observe that in adding three numbers, $a+b+c$, parentheses may be inserted in *two* ways, as: $(a+b)+c$ or $a+(b+c)$. Four numbers may be associated for addition in five ways. In how many different ways may parentheses be inserted in a sum involving five numbers? six? Can you find a general formula that gives the number of different ways that n numbers may be associated for addition—of course without changing the order of the numbers?

Let us illustrate how definitions are used to call attention to interesting concepts. Definitions in mathematics are no more than agreements to replace certain symbols, whose meaning is understood, by other symbols. These new symbols are to have precisely the same meaning as the old. Definitions usually enable us to express mathematical ideas more concisely, but the real test for a definition is whether it can be used as a tool in proofs.

***Definition 1:** When we write "$a>b$," we mean that there is a number c in S with $c\neq 0$ such that $a=b+c$. We read this new symbol as "a is greater than b."*

Theorem 4: If $a\neq 0$ then $a>0$.

Proof: $a=0+a.$ — A 3, A 1 and Def. 1

Theorem 5: $a>b\rightarrow a+c>b+c$.

Proof:

$a>b\rightarrow a=b+t$ with $t\neq 0$ — Def. 1

$a=b+t\rightarrow a+c=(b+c)+t$ — A 1, A 2

$a+c=(b+c)+t\rightarrow a+c>b+c.$ — Def. 1

Theorem 6:	$a+c>b+c \rightarrow a>b.$	
Proof:	$a+c>b+c \rightarrow a+c=(b+c)+t$	Def. 1
	$a+c=(b+c)+t \rightarrow a+c=(b+t)+c$	A 1, A 2
	$a+c=(b+t)+c \rightarrow a=b+t$	A 4
	$a=b+t \rightarrow a>b.$	Def. 1

Definition 2: When we write "$x<y$" we mean $y>x$, and we read "$x<y$" as "x is less than y."

Theorem 7:	$x<y$ and $y<z \rightarrow x<z.$	
Proof:	$x<y \rightarrow y=x+a$	Def. 1, 2
	$y<z \rightarrow z=y+b$	Def. 1, 2
	So, $z=(x+a)+b=x+(a+b)$	A 2
	And, $z>x$, so $x<z.$	Def. 1, 2

Definition 3. When we write "$a-b=c$" we mean $a=b+c$. We read this new symbol as "a minus b is c." The process of determining c when a and b are known is called subtraction. If $b>a$ the expression "$a-b$" is assigned no meaning.

Theorem 8:	If $x-y=z$ then $x-z=y.$	
Proof:	$x-y=z \rightarrow x=y+z$	Def. 3
	$x=y+z \rightarrow x=z+y$	A 1
	$x=z+y \rightarrow x-z=y.$	Def. 3
Theorem 9:	If $y<x$ then $(x-y)+y=x.$	
Proof:	Set $x-y=t$, then $x=y+t$	Def. 3
	Hence, $(x-y)+y=t+y=y+t=x.$	A 1
Theorem 10:	$(a+b)-b=a.$	
Proof:	Obvious by Def. 3 and A 1.	
Theorem 11:	If $s<r$ than $r-(r-s)=s.$	
Proof:	$r=(r-s)+s$	Theorem 9
	$r-(r-s)=s.$	Def. 3
Theorem 12:	If $a+b<c$ then $c-(a+b)=(c-a)-b.$	
Proof:	$c=(a+b)+t=a+(b+t)$	Def. 2, A 2
	$c=a+(b+t) \rightarrow c-a=b+t$	Def. 3
	$c-a=b+t \rightarrow (c-a)-b=t$	Def. 3
	$c=(a+b)+t \rightarrow c-(a+b)=t$	Def. 3
	So, $c-(a+b)=(c-a)-b.$	
Theorem 13:	$b<a$ and $c<b \rightarrow (a-c)-(b-c)=a-b.$	
Proof:	$(a-c)-(b-c)=a-(c+(b-c))$	Theorem 12
	$a-(c+(b-c))=a-b.$	A 1, Theorem 9
Theorem 14:	$a-a=0.$	
Proof:	$a=a+0.$	A 3, Def. 3
Theorem 15:	$b<a$ and $c<b \rightarrow a-(b-c)=(a-b)+c.$	
Proof:	$(a-(b-c))+(b-c)=a$	Theorem 9
	$((a-b)+c)+(b-c)=(a-b)+(c+(b-c))$	A 2
	$(a-b)+(c+(b-c))=(a-b)+b$	A 1, Theorem 9
	$(a-b)+b=a$	Theorem 9
	So, $(a-(b-c))+(b-c)=((a-b)+c)+(b-c)$	
	And, $a-(b-c)=(a-b)+c.$	A 4

The numerical examples below illustrate some of these theorems:

1. $(987-653)+653=$	Theorem 9
2. $6423-(6423-854)=$	Theorem 11
3. $97-43=(97-40)-3=$	Theorem 12
4. $(5000-2416)-(4000-2416)=5000-4000=$	Theorem 13
5. $90-59=(90-60)+1=$	Theorem 15

Definition 4. If $a=bc$ and $b\neq 0$ we write "$a\div b=c$" and say "a divided by b equals c." The process of determining c when a and b are known is called division. If $a\div b=c$ we say "b divides a" and write this as "b/a."

Theorem 16:	If $a\div b=c$ and $c\neq 0$ then $a\div c=b$.	
Proof:	$a\div b=c\rightarrow a=bc\rightarrow a=cb\rightarrow a\div c=b$.	Def. 4, M 1
Theorem 17:	If y/x then $(x\div y)\cdot y=x$.	
Proof:	Set $x\div y=t$, then $x=yt$.	Def. 4
	Hence, $(x\div y)\cdot y=ty=yt=x$.	M 1
Theorem 18:	$ab\div b=a$.	
Proof:	Obvious by Def. 4 and M 1.	
Theorem 19:	If s/r then $r\div(r\div s)=s$.	
Proof:	$r=(r\div s)\cdot s$	Theorem 17
	$r\div(r\div s)=s$	Def. 4
Theorem 20:	If ab/c then $c\div(ab)=(c\div a)\div b$.	
Proof:	$c=(ab)t=a(bt)$	Def. 4, M 2
	$c=a(bt)\rightarrow c\div a=bt$	Def. 4
	$c\div a=bt\rightarrow(c\div a)\div b=t$	Def. 4
	$c=(ab)t\rightarrow c\div ab=t$	Def. 4
	So, $c\div ab=(c\div a)\div b$.	

The reader may have noticed a curious duality between the set of Theorems 9, 10, 11, 12 and the last four, 17, 18, 19, 20. This is the type of duality encountered in projective geometry and Boolean algebra. We restate some of these theorems for comparison:

Theorem 9: $y<x\rightarrow(x-y)+y=x$.
Theorem 17: $y/x\rightarrow(x\div y)\cdot y=x$.

Theorem 11: $s<r\rightarrow r-(r-s)=s$.
Theorem 19: $s/r\rightarrow r\div(r\div s)=s$.

Theorem 12:
$a+b<c\rightarrow c-(a+b)=(c-a)-b$.
Theorem 20:
$ab/c\rightarrow c\div(ab)=(c\div a)\div b$.

This duality extends to the proofs of these theorems. That is, if in the proof of Theorem 12 the signs "$<$," "$-$," "$+$" are replaced by "$/$," "$\div$," "$\cdot$" respectively, then the proof of Theorem 20 results. The reader may now construct and prove the dual theorems on division corresponding to Theorems 13, 14, and 15.

The following list of exercises offers opportunities to use the concepts that have been developed.

Exercises

1. $(20-(20-12))-(12-7)=$
2. $(80\div(80\div 16))\div(16\div 2)=$
3. Is it true that for every pair of numbers a, b with $a>b$, we have $(a-b)+(2b-a)=b$?
4. Show that if $(a-b)=(c-b)+t$ with $t>0$ then $a>c$.
5. Show that if $a>b$ then $a-(a-(a-(a-b)))=b$.

Determine whether or not the following equations have solutions in S. We remark

that to *solve* an equation relative to the set S means to indicate all numbers in S that make the equation a true statement.

6. $15-(5-x)=5x-(5-x)$
7. $12-(x-4)=4x-(x-4)$
8. $2x-(2x-4)=2x-20$
9. $(t+1)\cdot 5=0$
10. $(3t+8)-5t=0$
11. $(3a-30)\cdot(40-5a)=0$
12. $2r+(2r-7)=2r$
13. $(5s+4)\div(3s-8)=5s+4$
14. $(s-2)\div 5=s-2$
15. $x-(x-(x-4))=x-4$
16. $(8-x)+x=8$

Are these statements true or false?

17. There is no number x in S such that $3(x-4)=2x-12$.
18. If $3x-(x+y)=12-y$ then $x=6$.
19. If x, y, $x-y$, $2y-x$ and $4-x$ are all natural numbers then $x=3$ and $y=2$.
20. If $2x-3<8$ and $3x-4>9$ then $x=5$.
21. If $ab=0$ then either $a=0$ or $b=0$.
22. For all numbers a, b, c in S with $b>c$ it is true that $a(b-c)=ab-ac$.
23. The only root of the equation $(t-2)(3t-9)=0$ is the number 3.
24. $8/4$.
25. a/b and $b/a \rightarrow a=b$.
26. $0/4$.
27. $2/(6a+4)$ for every a in S.
28. If b is any number in S then $2/b(b+1)$.

Complete the following statements.

29. If $xyz=t$ then $t\div xz=$ . . . .
30. If $x\div t=4$ and $y\div t=3$ then $(x+y)\div t=$ . . . . .and $xy\div t=$ . . . .
31. If $a\div b=27$ and $c\div b=9$ then $(a+2c)\div b=$ . . . . and $(a-c)\div b=$ . . . . and $3ac\div b=$ . . . . .and $a\div c=$ . . . .
32. If b/a and $b=a-1$ then $a=$ . . . . . . and $b=$ . . . .
33. If x/y and $x<y$ and $3x>y$ then $y\div x=$ . . . .
34. If $rs\neq 0$ then $5rrs\div rs=$ . . . .
35. If $3/ab$, and if the statement, $3/a$, is false then . . . .
36. If $a\neq 1$ and $b\div a=b$ then . . . .

We comment upon a few of the problems above.

The statement in problem 3 is false. To see this take $a=10$, $b=2$ and remember that we are dealing only with numbers in S. Statements about numbers are always made relative to a specified number system. A statement may be true when it refers to the integers and false when it refers to fractions. As an example, consider: *There is no number x greater than 3 and less than 4.*

The equations in (7), (9), (12) have no solutions in S. The equation in (11) has no solution. The equation in (16) has nine solutions. The statements in (17) and (23) are true.

Letter to the editor

May 23, 1957

Dear Sir:

With reference to the note entitled: "When is Easter?" which appeared on page 310 of the April issue of The Mathematics Teacher, perhaps some readers might be interested in a more extensive article on the subject. I refer to: "The Mathematics of Easter" by Mr. E. J. F. Primrose which appears on pages 225 ff. of the December, 1951, issue of the British publication: *The Mathematical Gazette*, vol. 35, no. 314.

In any given year, Easter is determined to be the first Sunday after the Paschal full moon, which is the first full moon after the 21st of March. Gauss's formula may easily be broken down into two parts, one of which gives the date of this full moon, and the other, the number of days between this date and the following Sunday. Mr. Primrose compares the Gaussian formula with the formula given in the Book of Common Prayer and also with the actual date of the full moon, calculated theoretically. These do not all agree in certain years, as the interested reader may learn for himself.

Sincerely,
Frank T. Gutmann
120 Highland Avenue
Auburn, Maine

Mathematics in general education[1]

W. I. LAYTON, *Stephen F. Austin State College, Nacogdoches, Texas.*

A survey of requirements and recommendations for mathematics in college general education programs in the United States.

NUMEROUS STUDIES HAVE been made on the mathematics that should be included in the general education program at the college and university level. Only two will be cited here—"The Content of a Course in General Mathematics—Teachers' Opinions"[2] and "College Mathematics for Non-Science Students."[3] Both of these studies are quite significant. However, we are looking for more recent data than those contained in the article by Brown, and the California study is neither as wide in scope nor does it use the same approach as the study described in the present paper.

The purpose of the present study is to survey the mathematical requirements of colleges throughout the nation that have a general education program and to analyze the opinions of representatives of these colleges and other colleges interested in general education as to the mathematical training most desirable for college students who enroll in the general education programs.

In order to collect data, a questionnaire to be completed by the head of the department of mathematics or a member of his department designated by him was sent in September, 1956 to a wide distribution of colleges. The mathematics courses required for various specialized curricula were not included. The questionnaire was mainly concerned with the mathematical needs of students enrolled in the general education program.

The questionnaire was divided into four sections: (1) general questions concerning the individuals and institutions responding; (2) entrance requirements in mathematics; (3) present requirements in college mathematics; and (4) recommendations concerning college mathematics.

It was found that in each of the forty-two states participating in the survey there were usually one, two, or three institutions that responded. The greatest number of replies from any one state was fourteen. Also of interest as we examine the data are the enrollment figures of the 150 colleges participating in the study. Seventy-seven of them have fewer than 2500 students, thirty-one between 2500 and 5000, and thirty-five schools have 5000 students or over. Seven of the colleges did not respond to the question on enrollment.

Some of the colleges are tax supported while others are privately endowed and/or church-related institutions.

It can be seen from the foregoing data that this study is virtually nationwide in scope and represents institutions of a wide variety of sizes and interests.

Also under the heading of general information was the question, "Do you have

[1] Based upon a paper presented at the Seventeenth Christmas Meeting of the National Council of Teachers of Mathematics at Jonesboro, Arkansas, December 26–29, 1956.

[2] Kenneth E. Brown, "The Content of a Course in General Mathematics—Teachers' Opinions," THE MATHEMATICS TEACHER, XLIII (January 1950), 25–30.

[3] "College Mathematics for Non-Science Students," a report of the Special Committee on College Mathematics for Non-Science Students, a sub-committee of the California Committee for the Study of Education, *The American Mathematical Monthly*, LXIII (November 1956), 639–642.

a general education program in your institution?" Ninety-one of the 150 colleges in the survey responded "yes" and fifty-nine "no." It is interesting to observe the size of these ninety-one institutions that have a general education program. Fifty-one of them have fewer than 2500 students, eighteen have between 2500 and 5000 students, while nineteen have an enrollment of 5000 students and over. Three of the ninety-one schools did not give figures on enrollment. We notice here that 56 per cent of the colleges included in this study that have general education programs have fewer than 2500 students. This indicates, perhaps, a trend for the smaller college to include a general education program in its curriculum. We also notice that of the remaining schools approximately one-half have between 2500 and 5000 students and the other half 5000 students and over.

The question was now raised, "If you do have a general education program, what student groups enroll in this program?" Seventy-five of the ninety-one colleges having a general education program require their liberal arts students to follow it. Seventy-four of the colleges with a general education program make this program mandatory for their teacher education candidates. Twenty-seven colleges require all students to enroll in the general education program. Apparently, liberal arts and teacher education are the two main areas where general education programs are, for the present at least, required.

Let us consider now the entrance requirements in mathematics. The question was asked, "How many high-school units in mathematics are required for entrance for your freshmen who will take the general education program and not take any mathematics in college unless it is general mathematics or its equivalent?" The ninety-one colleges answering this question have requirements for freshmen who will take the general education program ranging from zero to three high school units in mathematics for entrance. Only seventeen of the schools specify certain courses in mathematics for entrance and the courses specified are usually algebra I and/or geometry I.

Present requirements in college mathematics were dealt with next in the questionnaire. The question was asked, "Do you now require mathematics in your general education program?" Required mathematics here referred to the mathematics required of all students involved in the general education program except those students excused from mathematics by some institutional policy. Forty-six colleges or slightly over 50 per cent of those in the study having a general education program require some mathematics in this curriculum.

The colleges that require mathematics in their general education programs were asked to name the courses and the hours required. Twenty-nine of the forty-six institutions concerned named general mathematics as the course required in this program. Among the courses added by the respondents as being required in their institutions were: introduction to college mathematics, fundamentals of mathematics, elementary mathematical analysis, and college arithmetic. In some cases general mathematics was given as an alternate for college algebra or trigonometry. In one instance, general mathematics or history of mathematics or any other mathematics through the first course in calculus will suffice. Apparently, general mathematics predominates as far as present requirements are concerned in the general education programs.

In the ninety-one schools that have general education programs the mean of required mathematics is 1.97 semester hours. In computing this mean, zeros were used for those colleges that require no mathematics in the general education program. The range in required semester hours is from zero to eight.

The last question in the section "Present Requirements in College Mathematics" was, "If you excuse general edu-

cation students from taking general mathematics or its equivalent, what basis do you use?" Seven of the forty-six colleges that require mathematics in their general education programs use the number of high-school units in mathematics presented as a criterion for exempting their general education students from taking mathematics. Eleven of the schools exempt students on the basis of entrance tests while eighteen will permit other mathematics courses in college to be substituted for general mathematics. It is easily possible that a number of the schools that use high school units as a criterion for excusing people from the general education mathematics also require a satisfactory performance on one or more entrance tests in mathematics. No one of the above three practices is used by a highly significant number of institutions.

Mathematics courses that may be substituted for general mathematics in some of the college general education programs are: intermediate algebra, practical trigonometry, mathematics of finance, foundations of geometry, two courses in the sequence of specialized mathematics, any course of higher rank, traditional courses, commercial arithmetic, and plane geometry. Each of the above substitutions is allowed by at least one college. Two of the colleges will permit any college mathematics course to be substituted while four name college algebra and four specify trigonometry as suitable substitutions.

The final section of the questionnaire concerned recommendations in college mathematics. The recipients of the questionnaire were asked to react to the questions in this section in terms of what they felt was desirable in the way of mathematical training for students in the college general education program. It was explained that students who would pursue specialized mathematics such as business mathematics, mathematics for engineers, etc., were not to be included. The first question in this part of the questionnaire was, "What mathematics do you think should be required in the general education program?" General mathematics received a preponderance of votes with a total of ninety-three out of the 127 cast on this question. Thus, 73 per cent of the schools favor general mathematics in the general education program. College algebra and trigonometry came next although they rated considerably lower on the scale, receiving eight and six votes respectively. Other courses were added by the respondents as follows, although in no case did any of these receive more than two votes: integrated course; consumer mathematics; elementary point sets; introductory calculus; set theory and modern elementary probability; analytical geometry; calculus; elementary statistics; mathematics for liberal arts students; intermediate algebra; applied mathematics; fundamentals and foundations of mathematics; remedial and functional mathematics; elementary mathematical analysis; mathematics in human affairs; fundamental concepts of mathematics; and universal mathematics. Thus, while general mathematics is predominantly favored, we find a wide variety of other courses recommended to serve its purpose.

The question that followed asked how many hours of mathematics should be required in the general education program. It is interesting to find that the mean of all mathematics that the respondents think should be required in the general education program is 4.82 semester hours. This, of course, includes the colleges that would require no mathematics in the general education program. This mean of 4.82 semester hours in mathematics that was recommended is almost two and one-half times the mean of mathematics presently required. As pointed out previously, the mean of required mathematics in colleges that have general education programs is 1.97 semester hours.

The final portion of the questionnaire contained a list of thirty-six topics that might be included in college general mathematics. Those respondents who favored

general mathematics for a general education program were asked to vote whether each topic was highly desirable, desirable, or whether it should be omitted from a course or courses in general mathematics. Those answering the questionnaire were also given the opportunity to indicate any other topics that they felt were important.

Twenty-four topics were declared desirable or highly desirable in general mathematics by 50 per cent or more of the 128 colleges expressing an opinion in this matter. Following is a list of these twenty-four topics arranged in the order of their popularity, beginning with those receiving the greatest number of votes: graphs; the nature of proof; significant digits; introduction to algebra; whole numbers; mathematics as a language; fractions and mixed numbers; ratio and proportion; powers and roots; bases other than ten; decimals; per cents; mathematics as a science; scientific notation; mathematics as a tool; measures of central tendency; mathematics as an art; systems of measure; properties of geometric figures; logarithms; lines, angles, and planes; insurance; topics from analytic geometry; and solution of triangles.

A number of other topics were suggested by the respondents but in no case was there any high degree of agreement and no suggested topic received more than four votes.

Summary

In this study we have attempted to survey the mathematical requirements of a representative group of colleges offering programs in general education. We have also tried to analyze the opinions of spokesmen for these colleges and other colleges interested in general education as to the mathematical training most desirable for college students who are pursuing general education courses of study.

As a means of gathering the data a questionnaire was sent into the field. The questionnaire was in the main concerned with the mathematical needs of students enrolled in the general education program. The mathematics courses required for specialized curricula were not included.

The four sections of the questionnaire dealt with general information, entrance requirements in mathematics, present requirements in college mathematics in the general education program, and recommendations concerning college mathematics in this program.

One hundred fifty colleges of varying size and type in forty-two states participated in the survey which seems, consequently, to have considerable nationwide significance. Of these 150 colleges, ninety-one have a program of general education. An analysis of the size of these institutions seems to indicate a trend for the smaller college to include a general education program in its curriculum. Furthermore, liberal arts and teacher education are the two main areas where general education programs are required.

The entrance requirements in mathematics for freshmen who enroll in the general education program in the ninety-one colleges that have such programs range from zero to three high-school units in mathematics. These students will not take any mathematics in college unless it is general mathematics or its equivalent. Only seventeen of these ninety-one schools specify certain courses in mathematics for entrance, and the courses specified are usually algebra I and/or geometry I.

It is interesting to find that forty-six colleges, or slightly over 50 per cent of those having a general education program, require some mathematics in this curriculum. General mathematics was listed by twenty-nine of the institutions as a requirement. Thus, of the schools requiring mathematics in their general education programs, over 50 per cent specify general mathematics. The mean of required mathematics in the ninety-one colleges that have general education programs is 1.97 semester hours. The range is from zero to eight semester hours.

The criteria most frequently used in excusing general education students from

taking general mathematics or its equivalent were: the number of high school units of mathematics presented for entrance; entrance tests; and other college mathematics to be substituted for general mathematics. In this matter of substituting it was found that a rather wide variety of mathematics courses may be taken in place of general mathematics in the college programs of general education.

Under the heading of college mathematics recommended for students in the college general education program, general mathematics received 73 per cent of the votes. It was made clear, of course, that students who would pursue specialized mathematics such as business mathematics, mathematics for engineers, etc., were not to be considered as prospects for general mathematics. The remaining 27 per cent of the colleges favoring college mathematics as part of the general education program listed quite a wide variety of courses that could be taken.

The mean in semester hours of mathematics that the institutions felt should be required in the general education program was 4.82 semester hours. This mean is more than twice the mean of 1.97 semester hours of mathematics required in the colleges having general education programs.

The final portion of the questionnaire concerned a list of topics that might be included in general mathematics. Twenty-four topics were declared desirable or highly desirable in general mathematics by 50 per cent or more of the 128 colleges that reacted to these topics. These twenty-four topics could be classified mainly as algebra and arithmetic. A few of the topics are frequently treated in analytic geometry, plane geometry, statistics, and trigonometry.

According to the present study it seems evident that general mathematics is the course both most frequently required at present and most highly recommended for the general education program in colleges and universities in the United States.

Measurement

A yardstick measures length;
Your age it will not tell;
It's no thermometer
To see if you are well.

Pints don't measure length,
Nor tell your width or height;
They measure not how far;
They tell not dark or light.

A ruler measures size
In inches or in feet.
Would it serve to measure eggs
And oranges, apples, meat?

Speedometers tell speed
While riding in a car.
Would they tell you truly
The distance to a star?

You measure things each day,
Big and little, short and tall,
How much, how near, how far.
One stick won't measure all.

—*Taken from Peter Lincoln Spencer and Marguerite Brydegaard,* Building Mathematical Concepts in the Elementary School *(New York: Henry Holt and Company,* 1952), *p.* 57. *Reprinted with the permission of the authors.*

What is a geometric tangent?

RICHARD V. ANDREE, *University of Oklahoma, Norman, Oklahoma.*

Is your definition adequate?

Do YOU KNOW the meaning of the phrase "line tangent to a curve"? Before you read on, take a minute to write down your definition of a "line tangent to a curve." Your definition will be an important part of the article, so please write it out now.

Before considering a definition which has been found quite satisfactory, let us examine some of the pseudo-definitions which students often suggest.

Pseudo-definition 1: "A tangent is a line perpendicular to the radius (at its extremity)."

Even in the case of a circle, the definition is inadequate since the line segment "radius" has two extremities. This difficulty can be overcome, but the definition is still inadequate since it is valid only for circles. (Readers who dimly recall a course in calculus may grasp at a straw called "radius of curvature." This is inadequate, since the definition of radius of curvature involves the idea of tangent line, and circular reasoning results.)

Pseudo-definition 2: "A tangent is a line which touches the curve at only one point."

The sketches on this page demonstrate that this description is inadequate:

Figure 1

Line L touches curve C at only one point P, but L is *not* a tangent line.

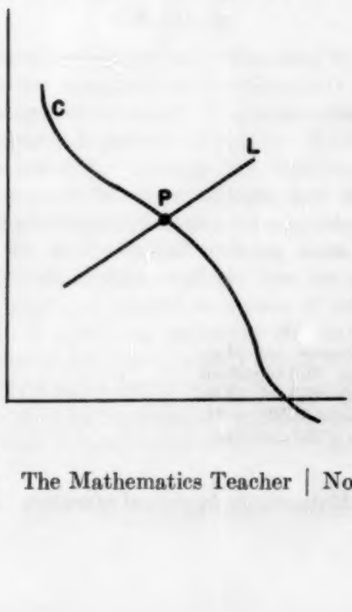
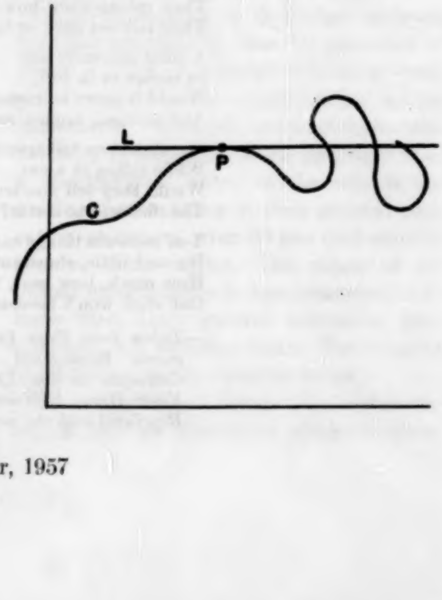


Figure 2

Line L is a tangent to curve C at point P, but it also crosses curve C at at least three other points.



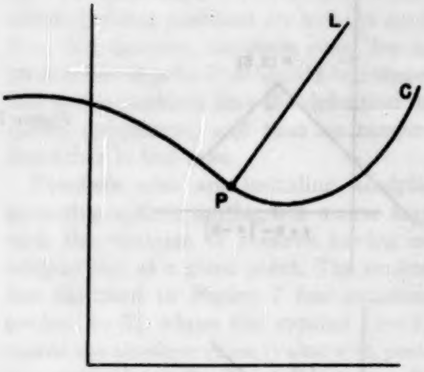


Figure 3

Line segment *L* touches curve *C* at point *P*, but does not cross it; however, *L* is *not* tangent to *C* at *P*.

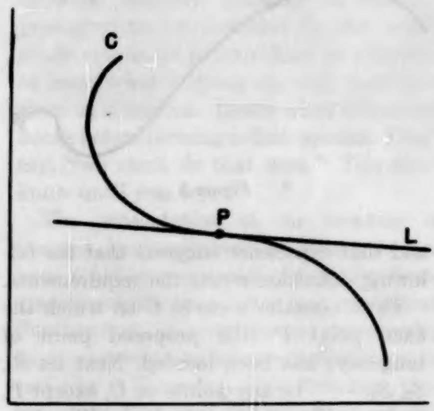


Figure 4

Line *L* is tangent to curve *C* at point *P*, but *L* also crosses *C* at *P*.

Pseudo-definition 3: "A tangent is a line which touches the curve, but does not cross it."

Figures 3 and 4 indicate that the proposed statement does not provide an adequate definition. Furthermore, the statement involves the ungeometric concept of "touching but not crossing," which can hardly be condoned in either synthetic or analytic geometry. Some readers will (rightly) object that the line segment terminating at P in Figure 3 is not a line, but a half line. However, Figure 4 shows a line which is tangent to the curve, and also crosses the curve *at the point of tangency.*

Pseudo-definition 4: "A line L is tangent to a curve C at a point P if, and only if, P lies on both L and C and it is possible to construct a small circle with center P such that C and L have no point in common inside the circle except the point P."

This seems a bit complicated, but if it were valid, it would not be too complicated to be useful. However, it too is neither necessary nor sufficient. It is clearly not sufficient, since any line crossing the curve (Figure 1) satisfies pseudo-definition 4. Furthermore, the condition is not even necessary, although a counterexample is somewhat more difficult to arrange. The curve consisting of that portion of $y=\sin\ (\pi/x)$ which lies between $x=-1$ and $x=1$, combined with the half circle $y=+\sqrt{1-x^2}$, provides one counterexample, since the line $y=1$ is tangent to the curve at (0, 1), but in any small circular region about (0, 1) the line $y=1$ and the curve $y=\sin\ (\pi x)$ have infinitely many points in common. Other counterexamples may also be found. However, this is unnecessary since Figure 1 proves the definition inadequate.

In ordinary classroom teaching, the author usually has the students suggest possible definitions, and then gives counterexamples to show why the suggested definitions of "line L tangent to a curve C" are inadequate (which they usually are). After perhaps ten minutes the class is about ready to concede that perhaps they are not able to define "line tangent to a curve," although most of them still feel (perhaps justifiably) that they know what one is. This is a crucial point at which it is possible to introduce a valid definition. It is my custom to explain to the students that, before continuing with the study of mathematics, it is necessary to have a valid definition of tangent line, since much of the work in mathematics (I avoid the word calculus) depends upon this concept,

Figure 5

and that experience suggests that the following definition meets the requirements.

First, consider a curve C on which the fixed point P (the proposed point of tangency) has been located. Next let S_1, S_2, S_3, $\cdots$ be any points on C, except P, and consider the secant lines PS_1, PS_2, PS_3, PS_4, $\cdots$.

If the sequence of points S_1, S_2, S_3, S_4, $\cdots$, each distinct from P, is so chosen (on C) that the sequence of *arc lengths*, PS_1, PS_2, PS_3, PS_4, $\cdots$ approaches length zero [i.e., $\lim_{n\to\infty}$ (length of arc PS_n) $=0$], then the sequence of secants PS_n will have the tangent line at P as limiting position, if such a tangent line exists.

Definition: The tangent line PT to a curve at a fixed point P on the curve is the limiting position of the sequence of secant lines PS_n, where the points S_n are so chosen on the curve that the sequence of (lengths of arcs PS_n) approaches zero, *providing this limiting position exists and is unique.*

After some boardwork with a ruler, it is time to consider the possibility of a (continuous) *curve which fails to have a tangent* at a given point on the curve. Figure 7 serves admirably.

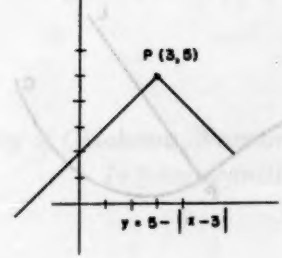


Figure 7

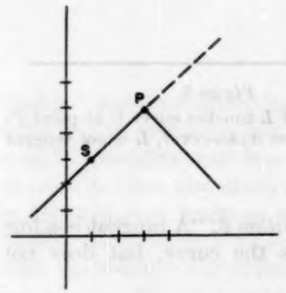


Figure 8

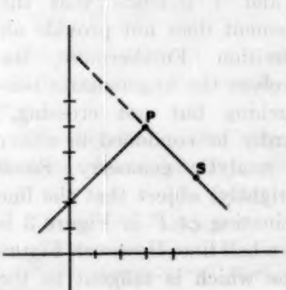
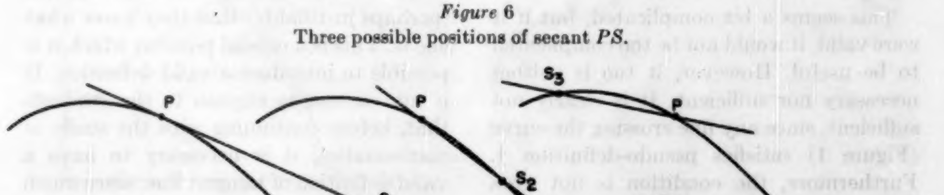


Figure 9

Figures 8 and 9 show two possible positions of S, and the dotted lines show the (one-sided) limiting positions of secants through P which contain points S; on the

Figure 6
Three possible positions of secant PS.



curve on each side of P. Since the two (one-sided) limiting positions are not the same (i.e., not unique), *the given curve has no tangent line at point P.* It should be pointed out to the students that the definition requires uniqueness, and that no tangent line exists in this case.

Teachers who are including analytic geometry in their tenth-grade course may wish the equation of a curve having no tangent line at a given point. The broken line sketched in Figure 7 has equation $y=5-|x-3|$ where the symbol $|x-3|$ means the absolute value (value with positive sign) of $(x-3)$. Point P has co-ordinates (3, 5).

If you have not considered the many advantages of including co-ordinate (analytic) geometry in your tenth-grade class, you should do so. Many teachers are enthusiastic. The current New York State Syllabus specifies portions of analytic geometry to be included in the tenth-grade course. It is your duty as a teacher to learn what is going on, and then form your own opinion. Learn what others are doing before forming a firm opinion. Don't say, "we can't do that here." You don't know until you try.

The presentation of the meaning of tangent line suggested in this note has been tried in numerous classes. Students grasp it readily once it has been explained. Furthermore, it provides an opportunity to discuss what an adequate definition is, and why it is needed. This definition provides the first directed thinking on the matter of limits which the student encounters. Don't handicap your students by failing to discuss geometric limit.

The author invites correspondence from interested teachers.

What's new?

BOOKS

SECONDARY

Algebra, Second Course, John R. Mayor and Marie S. Wilcox, Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957. Cloth, vi+458 pp., $3.92.

COLLEGE

Calculus with Analytic Geometry, Richard E. Johnson and Fred L. Kiokemeister, Boston: Allyn and Bacon, Inc., 1957. Cloth, xi+650 pp., $7.95.

Understanding Arithmetic, Robert L. Swain, New York: Rinehart and Company, Inc., 1957. Cloth, xxi+264 pp., $4.75.

MISCELLANEOUS

Applied Probability, Proceedings of Symposia in Applied Mathematics, Volume VII, New York: McGraw-Hill Book Co., Inc., 1957. Cloth, v+104 pp., $5.00.

Mathematics for Everyman, Egmont Colerus (translated by B. C. and H. F. Brookes), New York: Emerson Books, Inc., 1957. Cloth, xi+255 pp., $3.95.

The Logical Problem of Induction (revised edition), Georg Henrik von Wright, New York: The Macmillan Company, 1957. Cloth, xii+249 pp., $4.00.

BOOKLETS

Arithmetic, Algebra, Logarithms and Slide Rule, Robert L. Erickson, 2724 Waunona Way, Madison, Wisconsin. 93 pp., $2.00.

Mathematics Teaching Aids for a Stronger America, National Aviation Education Council, 1025 Connecticut Ave., N.W., Washington 6, D. C. Reprint of the Illinois Curriculum Program's Bulletin for an aviation education project, 75 pp., 75 cents.

You Can Be a Civil Engineer, American Society of Civil Engineers, 33 West 39th Street, New York 18, New York. 16-page illustrated booklet, free.

Providing for the student with high mathematical potential

FLORENCE L. ELDER, *Junior-Senior High School, West Hempstead, New York.*

***Enrichment** (with a capital* E) ***for the Gifted** (with a capital* G).
A seminar approach in the junior high school.

Who is the student with high mathematical potential, the gifted child? Where is he? How do we locate him? What shall we do with him after we have spotted him? These are pertinent questions that we dare not neglect.

All agree that the gifted child exists and that he is in our midst in the elementary school, in the junior high school, and in the senior high school. Since he has some of his most serious problems before he is twelve years old, it is of utmost importance that we locate him as early as possible.

We are often asked how the gifted child can be spotted in the classroom. Three interesting cases will be cited.

Early in the school year, Tom, a seventh grader, was referred to the department chairman because of his insistent reading of story books during mathematics class. Tom and his buddy were invited to the office for a visit. At this conference Tom was introduced to numbers written to bases other than 10. His interest was keen, and he became excited at the challenge of computing with these "new numbers." Within an hour he was actually performing "long division" in the duodecimal system.

Paul was also a seventh grader. He casually spoke of the binary system and its uses. In response to our questions he disclosed that he had read of the binary system two years earlier in the public library.

Steve was bored in his mathematics class. His grades were mediocre. The teachers in general complained of his careless work habits. However, he liked to "make up games, particularly baseball games."

Is it necessary to add that Tom, Paul, and Steve are highly gifted youngsters? But these are only instances. How do we recognize giftedness?

The gifted child often has unusual, perhaps odd, interests.

He has a rich associative background which is not to be confused with the social advantages possessed by some children.

He has quick understanding and a retentive memory.

He has insatiable curiosity. He is not satisfied in being told *How* to do a problem, but demands to know *Why*, seeking the proof as well as the reason.

He can discover and grasp several ways of solving the same problem. He will compare these methods and generalize his findings.

He has a longer attention span than the others in the class. He is interested in challenging units of work rather than short specific assignments.

He dislikes large amounts of routine and drill. He may neglect dull, practice homework assignments.

Unfortunately, there may be evidences of unsatisfactory adjustment to the school program, such as undesirable work habits, daydreaming and frequent idleness, poor grades and mediocre work, disrespectfulness and resentment of authority, intolerance of others, indifference and open distaste for school.

A unique experiment in the education of the gifted student has been operating in West Hempstead Junior-Senior High School for more than three years. At present seventh-grade and eighth-grade seminars are in operation. The program is an attempt to identify the gifted pupil at the outset of his junior-high career, to stimulate his intellectual curiosity and initiative, to extend the depth and scope of his curriculum, and to help him toward a better understanding of his associates and of his own responsibilities.

The selection of seminar members is the responsibility of a committee consisting of the principal, the guidance counselor, the district psychologist, and the various cooperating department chairmen. Students are screened and selected on the basis of standardized test results; teacher and guidance-counselor anecdotal reports, records, and recommendations; and the child's physical fitness, emotional maturity, and total behavior pattern as observed in a variety of situations. In the selection of seminar members the emphasis is on intellectual potential rather than on past performance. Standardized tests usually pick out gifted children very much better than do school marks. Since teachers tend to evaluate a child in terms of school achievement, they often fail in identifying the gifted.

During the screening process the prospective member meets several times with various members of the committee. The department chairmen discuss with these students the how's and why's of the operation of the seminar. At these conferences they are alert for evidences of the child's interest, curiosity, and ability to do relational thinking.

The organizational plan of the West Hempstead seminar program is unique. Students meet with their peers in all their regular classes and activities. They are, however, released from two of their regular classes to meet as a special group at semiweekly seminar sessions. Students thus excused are responsible for all work missed and must maintain a superior average.

The seminar classes are small. Each group is limited to fifteen members. Before classes commence, students who have been selected and their parents are notified by letter of the opportunity offered. At an informal meeting with the committee, the objectives and details of the program are discussed, and written parental consent is obtained.

Seminar classes have been conducted by the department chairmen. An enriched program of work, not included in the regular syllabus, is followed in the fields of English, history, mathematics, and science. Classes are conducted in an informal manner. Time is devoted to group discussions, committee study, laboratory workshops, library research, and individual studies and reports.

No text is used in the mathematics sessions. (A listing of references used may be found at the end of this article.) Pupils share in collecting facts, information, and ideas. They make use of libraries, industries, and other community agencies. The most valuable part of the mathematics course has been developed as a result of pupil experimentation and generalization.

Mathematics sessions are planned to awaken mathematical curiosity and interest as well as to provide a background for further study. The development of our number system is seriously considered. In the seventh grade this includes a historical approach. The student gains an understanding of the decimal system by investigating numbers written to other bases. He explores common fractions, "decimal"

fractions, and "per cent" within different number systems. He considers the relative advantages and disadvantages of operating with these numbers. Pupils who have discovered the relationships of "per quinage" (term coined by them), where $32/100 = .32 = 32\%$ (each to the base 5) equals the fraction 17/25 (in the decimal base), gain a deeper insight into the significance of percentage and decimal fractions.

Steve, the seventh grader whose specialty was originating games, developed an ingenious chart, "How to Change from One Base to Another," to avoid the detour of writing numbers to the base 10. A section of his paper will illustrate the creative thinking done within the seminar.

5 base to 8 base	1	2	3	4
2nd digit	5	12	17	24
3rd digit	31	62	113	144

This section of his chart will be used to convert the number 342, written to the base 5, to the corresponding number written to the base 8. The procedure is the addition, performed to the base 8, of numbers located in the table. The first addend is 2 because it is in the units place. Since the second digit in 342 is 4, the second addend is found in the table in the fourth column to be 24. To locate the third addend we note that the third digit is 3 which corresponds to 113 in the chart. The addition becomes

$$\begin{array}{r} 2 \\ 24 \\ \underline{113} \\ 141 \end{array}$$

The result may be checked by noting that 342 (base 5) and 141 (base 8) are both equal to 97 (decimal base).

Another example will illustrate further. The number 635, written to the base 7, will be written as the corresponding number to the base 3.

The procedure is addition performed to the base 3. The addends are 5, 210, and 101220. The sum is 102212 (written to the base 3). Hence the given number 635 (base 7) may be written as 102212 (base 3). The result may be verified by converting each to the base 10 (320).

Steve's chart has no practical application perhaps, but it has stimulated and quickened the mathematical imagination of those who have used it. In his regular mathematics class, Steve now does exceptional work.

Pupil experimentation and exploration is the keynote of the mathematics seminar. Not restrained or restricted by traditional outlines of study or exams, pupils continually amaze us with their "discoveries."

George's paper on "The Prime Number System" is another original contribution. His is a system for writing numbers:

1. with place value represented by the respective primes.

2. in which adjacent digits are multiplied.

3. with powers of numbers written by indicating the exponent of the prime in the respective place.

Thus, 11 in the Prime System represents 3×2, or 6, and 111 represents $5 \times 3 \times 2$, or 30. In this system, 11111 represents $11 \times 7 \times 5 \times 3 \times 2$ or 2310 (base 10), and 1011 indicates $7 \times 3 \times 2$, or 42 (base 10). Shall we write a few numbers in the Prime System and compare with their decimal system equivalents?

Prime System	*Decimal System*
1	2
10	3
2	4 (since 2^2 is 4)
100	5
1000	7
3	8 (since 2^3 is 8)
20	9 (3^2 is 9)
123	360 ($5 \times 3^2 \times 2^3$)

Addition and subtraction follow no set rules in this system, but multiplication

7 base to 3 base	1	2	3	4	5	6
2nd digit	21	112	210	1001	1022	1120
3rd digit	1211	10122	12110	21021	100002	101220

and division are relatively simple. The digits are added or subtracted as indicated by the operation. Thus:

Prime System	*Decimal System*
$111 \times 1 = 112$	$30 \times 2 = 60$ or $(5 \times 3 \times 2)(2) = 5 \times 3 \times 2^2$
$1010 \times 12 = 1022$	$21 \times 12 = 252$ or $(7 \times 3)(3 \times 2^2)$
$2101 \div 2000 = 101$	$490 \div 49 = 10$ or $(7^2 \times 5 \times 2) \div (7^2)$
Since, $1 \div 1 = 0$	$2 \div 2 = 1$

unity in the Prime System is defined as 0. The need for fractions is apparent in examples such as $100 \div 1$ or $100 \div 10{,}000$. (These are written in the Prime System.) Thus the system was extended to include fractions:

Prime System	*Decimal System*
.1, .01, .001, .11, .2	$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{4}$

This extension makes multiplication and division always possible. Division of fractions is performed as the inverse of multiplication. Simplification is readily done. To illustrate,

Prime System	*Decimal System*
$100 \div 1 = 100 \times .1 = 100.1$	$5 \div 2 = 5 \times \frac{1}{2} = \frac{5}{2}$
$20 \div 21 = 20 \times .12 = 20.12 = .1$	$9 \div 18 = 9 \times \frac{1}{18} = \frac{9}{18} = \frac{1}{2}$
Since $11 \div 11 = 11 \times 11 = 11.11$	$6 \div 6 = 6 \times \frac{1}{6} = \frac{6}{6}$
and also $11 \div 11 = 0$	$6 \div 6 = 1$

unity may be also written as 11.11, 1.1, 111.111, 10.01, 22.22, 12.21, or 0.0, as well as 0 in the Prime System. The resemblance of this system to logarithms is readily apparent.

"The Prime System is not important because it has any special merit, but because it illustrates certain universal properties of any number system. In the prime system we started with one operation, multiplication, but immediately had to put in the inverse operation, division. The number system wasn't complete until the operations were possible, and for that reason fractions were added." (Quoted directly from the paper.)

Investigation of George's number system by seminar members led to an elementary consideration of group theory, elements, operations, and postulates. Members were fascinated as they set up simple groups.

No school grade nor formal credit is offered for seminar studies. Membership is considered a privilege and opportunity. Throughout the year evaluation is made of the pupil's attitude, his achievement in his regular classes, and his personal response and adjustment to the program. The student shares in this evaluation.

At West Hempstead we are continually revising and implementing the seminar program. It has been planned in the light of our students' needs, the school's philosophy and facilities, and the available community resources. Students are contributing to the development of this program. At the close of each session they prepare a frank, unsigned appraisal. Their suggestions, evaluations, and recommendations are referred to the seminar committee for careful study. In June, 1956, questionnaires were submitted to all seminar members, their parents, classroom teachers, guidance counselors, and administrators. The report of this survey was the basis of the discussion at the June semiannual tea given for seminar members and their parents.

Pupils enjoy the seminar. They appreciate the relaxed atmosphere and the absence of tests and marks. The opportunity to do extended, guided research on a topic of interest has proved stimulating and rewarding to them. It was their unanimous opinion that the program is a help in their regular classwork because of the background provided. Students strongly recommended an extension of the program to include other capable seventh and eighth graders and a continuation of the program in the ninth year.

Parents expressed enthusiasm for the project. Their surveys indicate that very frequently students are stimulated to keener interest in school.

The released time of members from their regular classes has been a factor in the success of the program. During these periods the classroom teacher is able to spend additional time in developing difficult concepts without the risk of boring

the gifted child. Furthermore, seminar members show increasing maturity as they accept individual responsibility in making up all work missed during the released time.

Many have requested an extension of the seminar through the ninth grade. This has not yet been possible because of limited facilities. The seminar program for the gifted junior-high student at West Hempstead is still in the experimental stage. Early results indicate considerable success. Later followup studies will be made to measure the effectiveness of the project.

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The Mathematics Students' Journal.

Have you read?

ALLEN, FRANK B. "Mathematics Tomorrow," *N.E.A. Journal*, May 1957, pp. 309–310.

Mr. Allen, Chairman of the Secondary Curriculum Committee set up by the National Council of Teachers of Mathematics, has in this article pointed the way for the future. He is aware of the criticism leveled at mathematics, but this same criticism has brought about an awareness of the need for mathematics that was not felt in the past. He also points out how mathematics, as a fundamental way of thinking, is rapidly permeating other organized fields. This calls for the use of new mathematics principles which involves changes in the high school mathematics programs. He predicts the new program will include much more advanced work, that much of the artificial partitions will be eliminated, the less important will be de-emphasized, and differences in abilities will be given more consideration. This means there must be an improvement in the training of mathematics teachers, they must meet higher mathematics standards, and there will be closer articulation between methods of instruction and content.

These and several other items are discussed. This article is well worth thinking about.

ETTLINGER, H. J. "The Role of Mathematics in Every Day Life," *Scripta Mathematica*, June 1956.

Here is an article which, in a few short pages, gives one a feeling for the impact mathematics has had on the growth of the world. From the advance from one to two, which was a great step, to the main stream of progress today, we find mathematics the hard core. The author points out not only the practical uses to which mathematics is applied, but also the flowering of the intellectual ideas in themselves. He points out the stepped-up pace of mathematics and all progress in the last fifty years and brings out the need for learning in ways other than experience.

Through mathematics we have a liberalizing influence on the minds of men.

He goes so far as to say that through the alertness of mind which mathematics helps develop, we can guarantee the survival of our free way of life. Read it and see what you think.

WILLERDING, MARGARET F. "The Challenge of Practical Applications," *School Science and Mathematics*, June 1957, pp. 437–446.

Applications of the mathematics principles we study are a necessity and always a problem. "Practical" has so many different meanings. The author has taken a very realistic approach to the problem and gives some good advice as well as good illustrations. For example, applications must be real to the student, data must be live, students must be encouraged to develop problems on their own, the problems must challenge, and the results must be evaluated. She also gives some interesting illustrations on water sprinklers, projectiles, wire loads, and several other topics. I think this article will be very helpful in an area where we need to do more work.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Numerical analysis and high school mathematics[1]

YUDELL L. LUKE, *Midwest Research Institute, Kansas City, Missouri.*

At least one method now taught "should be relegated to the class of useless objects."

INTRODUCTION

THE LAST DECADE has witnessed a great upsurge in the application of mathematics and its discipline to the solution of problems pertaining to our physical and social environment. In the large sense, this research activity may be called applied mathematics. The mathematical expressions describing a physical model are usually so complicated that very little if any information can be gleaned from the equations themselves. To gain an understanding of the phenomenon, it is necessary to resort to computations. Here, computers and computer techniques play a dominant role. The art and science of computing is known as Numerical Analysis. It is concerned with effective and efficient methods and processes of obtaining numerical answers. The objective in Numerical Analysis is not only the results themselves, but also the development and study of methods used to obtain the results. The techniques must, above all else, be practical. Applied mathematics ultimately reduces to numerical data, and the automatic computer is a useful tool to obtain these numbers. Machines must be taught. Their intelligence level mirrors our own. Since its introduction, the automatic computer has generated revolutionary concepts and, if properly used, will breathe new life into many branches of the physical and social sciences.

In this paper we illustrate how Numerical Analysis, the art and science of computation, can be introduced in the high school mathematics curriculum. As a start, we are concerned with methods for the evaluation of a root (square root, cube root, and so on) of any number. This adventure leads to another, and off we go to Alice in Wonderland. Now practically every high school student learns the method for determining the successive digits in the square root of a number, and he may be surprised to learn that modern computing procedures never use the method taught in high schools. From the point of view of automatic computation, the technique is not very efficient, for it requires too many operations.

The ensuing discussion is somewhat abbreviated, for it is our hope that the material will reach students who will further develop the concepts treated. This approach should prove most rewarding, and expansion of the ideas presented along with solution of numerous suggested problems should prove suitable for an exhibit in a science fair.

SOME ELEMENTARY METHODS BASED ON ITERATIONS

The first method to be described employs an iterative procedure. Suppose that

[1] Based in part on a talk given at the National Council of Teachers of Mathematics meeting, in Jonesboro, Arkansas, December 28, 1956, and in part on an article by the writer previously published in "Mathematics Projects for High School Students," prepared by members of the Department of Mathematics of the University of Kansas, Lawrence, Kansas, for Science and Mathematics Day, October 29, 1955. This reference is out of print.

it is desired to compute $\sqrt{N}$. Any method is used to obtain a first approximation to $\sqrt{N}$. An educated guess will often do. This value is then used to obtain a better approximation, and the process is repeated until one achieves the desired accuracy. A complete discussion of the method must contain a proof that the successive numbers are better and better approximations.

It is obvious that $\sqrt{N}$ is the positive root of the equation

$$x^2-N=0. \tag{2.1}$$

Let x_0 be a first approximation to $\sqrt{N}$ and suppose that x_0+h is the exact value. Then

$$(x_0+h)^2-N=x_0^2+2x_0h+h^2-N=0. \tag{2.2}$$

It is obviously as difficult to compute the exact value of h as it is to compute $\sqrt{N}$. Hence, we look for a good approximation to h which is easy to determine.

If x_0 is a good approximation to N, then h is small, and h^2 is much smaller still. Thus, the equation for determining h is almost the same as

$$x_0^2+2x_0h-N=0. \tag{2.3}$$

We solve this equation for h (observe that it is a linear equation) and call the solution h_0. Thus,

$$h_0=\frac{N-x_0^2}{2x_0}. \tag{2.4}$$

Let

$$x_1=x_0+h_0=\frac{N+x_0^2}{2x_0}. \tag{2.5}$$

Thus, x_1 is a better approximation to $\sqrt{N}$ than x_0.

We can now repeat the entire process with x_1 in place of x_0 and h_1 in place of h_0. We obtain

$$h_1=\frac{N-x_1^2}{2x_1}, \quad x_2=\frac{N+x_1^2}{2x_1}.$$

This process can now be repeated indefinitely to obtain $x_1, x_2, \cdots, x_k, \cdots$, as a sequence of approximations to $\sqrt{N}$. If x_k is the kth approximation to $\sqrt{N}$, then we determine the next approximation by replacing

$$(x_k+h)^2-N=0$$

by

$$x_k^2+2x_kh_k-N=0$$

and solving for h_k. Thus,

$$h_k=\frac{N-x_k^2}{2x_k}, \quad x_{k+1}=\frac{N+x_k^2}{2x_k}. \tag{2.6}$$

Example: Let $N=5$, $x_0=2$. Then successive applications of formula (2.6) with $k=0, 1, 2, \cdots$, give

$$x_1=\frac{9}{4}=2.25$$

$$x_2=\frac{161}{72}=2.23611\ 11111$$

$$x_3=\frac{51841}{23184}=2.23606\ 79779.$$

The exact value of $\sqrt{5}$ to ten decimal places is 2.23606 79775. Comparison of x_2 and x_3 with the true value seems to indicate that the number of correct decimals is virtually doubled at each stage of the iterative process. Is a general statement possible?

The above iterative procedure can be speeded up in several ways. We mention two. From equation (2.6) we have

$$x_{k+1}=\frac{N+x_k^2}{2x_k},$$

$$x_{k+2}=\frac{N+x_{k+1}^2}{2x_{k+1}}.$$

Substitute the value of x_{k+1} from the first of these equations in the second. The resulting equation expresses x_{k+2} in terms of x_k. Thus, two cycles of the iterative procedure defined by formula (2.6) are now telescoped into one. Another trick is as follows. In equation (2.2), write h^2 as $h\cdot h$ and substitute for one of the h's, the value h_0 given by formula (2.4). The equation so derived is linear in h, and the resulting

solution can be used to define an iterative process.

It is possible to give a graphical explanation of the iterative scheme generated by equation (2.6). The graph of $y=x^2-N$ is a parabola, and the abscissas of the points where this curve crosses the x-axis are $\pm\sqrt{N}$. Consider the point (x_0, y_0). Take the tangent to the graph at (x_0, y_0). It can be shown that the abscissa of the point where this tangent crosses the x-axis is the second approximation x_1. Then take the point on the graph of $y=x^2-N$ whose abscissa is x_1; call this point (x_1, y_1). The tangent to the parabola at (x_1, y_1) crosses the x-axis at the point whose abscissa is the third approximation x_2. This process can be continued indefinitely.

METHODS BASED ON THE BINOMIAL SERIES

Again suppose that the problem is to compute $\sqrt{N}$. Let x_0 be a first approximation. Then

$$x=x_0\left(1+\frac{N-x_0^2}{x_0^2}\right)^{1/2}. \qquad (3.1)$$

Expand the expression on the right by the binomial series (when is this series convergent?). When the series converges, we can obtain $\sqrt{N}$ to any desired accuracy by taking a sufficient number of terms. If only two terms of the series are used, and if we designate x as x_1, the result leads to equation (2.5). If any finite number of terms is used in the series, it is possible to establish an iterative procedure similar to that given in (2.6).

Rather than solve equation (2.1), we might just as well try to solve

$$x^{2-k}-Nx^{-k}=0, \qquad (3.2)$$

where k is some number. The problem is this. Can we derive an iterative procedure based on the latter equation, and if so, how can we choose k to render the process most efficient? Note that the case $k=0$ has already been treated. Again let x_0 be a first approximation to $\sqrt{N}$, and suppose that x_0+h is the exact value. Substitute (x_0+h) in (3.2) and expand by the binomial theorem. Neglecting all terms beyond those involving h^2, we find after some manipulation that

$$(x_0^{2-k}-Nx_0^{-k})+h[(2-k)x_0^{1-k}+Nkx_0^{-1-k}]$$
$$+\frac{h^2}{2}[(2-k)(1-k)x_0^{-k}$$
$$-Nk(k+1)x_0^{-2-k}]=0. \qquad (3.3)$$

As before, we would like to have a linear equation to determine h. Note that the term involving h^2 is an approximation to the error committed by using only a finite number of terms in the binomial series. We can reduce the error if we force the coefficient of h^2 to vanish. By our assumption, x_0^2 is an approximation to N. In the coefficient of h^2, let us put $x_0^2=N$. Then if the coefficient of h^2 is zero, one finds that $k=\frac{1}{2}$. Now put $k=\frac{1}{2}$ in the first two terms of (3.3). Also replace h by h_0 and solve for h_0. Thus,

$$h_0=\frac{2x_0(N-x_0^2)}{3x_0^2+N}, \qquad (3.4)$$

and if $x_1=x_0+h_0$, we find

$$x_1=\frac{x_0(x_0^2+3N)}{3x_0^2+N}. \qquad (3.5)$$

The formation of an iterative process is now apparent. Do you recognize the latter two equations?

The student should compare the efficiency of the various procedures and investigate the advantages and disadvantages of each. Also, he should look for still other methods for approximating roots of numbers.

PROBLEMS

In the following, we list some problems in connection with the previous developments.

1. Compute $\sqrt{N}$ to ten decimal places for $N=0.1, 0.2, \cdots, 1.0, 2.0, 3.0, \cdots, 10.0$.

This is not intended as an arithmetic exercise, and a desk calculator will be helpful to carry out the numerical details.

Can this table be used to compute the square roots of other numbers?

2. In formula (3.3), put $k=2$ and neglect the coefficient of h^2. Derive an iterative process based on the resulting equation. What are its convergence properties? Does the scheme have computational advantages?

3. Need N be a real number? That is, what can you say about the various procedures if N is of the form $a+ib$ where a and b are real and $i=\sqrt{-1}$?

4. Give a complete treatment of the corresponding iterative procedures for computing the cube root of N, the fourth root of $N, \cdots$, the nth root of N.

5. Draw a large graph of $y=x^2-N$, and draw the tangents at the points (x_0, y_0), (x_1, y_1), and so on. It can be shown by using calculus and analytic geometry that the equation of the tangent to the parabola $y=x^2-N$ at the point (x_0, y_0) is $y-y_0=2x_0(x-x_0)$. Show that the abscissas of the points where these tangents cross the x-axis are the successive approximations $x_1, x_2, \cdots$, defined by equation (2.6).

6. The iteration method for approximating $\sqrt{N}$ can be generalized to a method for approximating the solution of almost any equation $f(x)=0$. We draw the graph of $y=f(x)$, and take x_0 as the first approximation to the solution of $f(x)=0$. Then draw the tangent to the graph $y=f(x)$ at (x_0, y_0), where $y_0=f(x_0)$. The abscissa of the point where this tangent crosses the x-axis is the second approximation x_1. The process can be repeated to obtain the further approximations x_2, $x_3, \cdots$. It is usually necessary to employ the calculus to find the equation of a tangent to a curve. Once the analytical formula for the tangent line is known, one can develop iterative procedures in equation form and so dispense with the need for graphing. Curve sketching is sometimes helpful to obtain a good approximation, but improved accuracy is best done with the aid of analytical formulas. This method for approximating the solution of an equation is known as Newton's method (after Sir Isaac Newton). It is also often called the Newton-Raphson process. Find an approximate solution of the equation $x-e^{-x}=0$. Hint: the equation of the tangent to the curve $y=x-e^{-x}$ at (x_0, y_0) is $y-y_0=(1+e^{-x_0})(x-x_0)$.

7. Suppose we are given a polynomial with real coefficients. Suppose further there is a real root and we know a good approximation. How can we use the concepts presented to get a better approximation? For example, suppose we require the real root of $f(x)\equiv x^3-9x^2+x+6=0$. Now $f(0)=6$ and $f(1)=-1$. Hence there is one real root at least between $x=0$ and $x=1$. Why? Sketch the curve $y=f(x)$. Does the sketch furnish a valid proof? We *suspect* that the root is nearer to 1 than to 0. Let us digress. If $f(a)=A>0$ and $f(b)=B<0$, and if B is smaller in magnitude than A, it is *not* necessarily true that a root of $f(x)=0$ is closer to b than to a. Can you construct some examples? To return to our original course, we continue with our hunch that a root is near 1. Let $x_0=1$. Substitute $x=x_0+h=1+h$ in the equation for $f(x)$. Discard all powers of h higher than the first and replace h by h_0. Thus,

$$(1+h)^3-9(1+h)^2+(1+h)+6=0$$

leads to

$$1+3h_0-9(1+2h_0)+(1+h_0)+6=0$$

and so

$$h_0=-\frac{1}{14}$$

whence

$$x_1=x_0+h_0=13/14$$

is a better approximation. This again is the Newton-Raphson process and can be repeated over and over to obtain values that are closer and closer to the true root. The Newton-Raphson process usually works quite well, though it is not a panacea. Can you deduce limitations of the process? How do you get around the limitations?

The above calculations can be obtained by the method of synthetic division. We first divide $f(x)$ by $x-1$. In the form of detached coefficients we have

$$\begin{array}{r|l} 1-9+1+6 & 1 \\ 1-8-7 & \\ \hline 1-8-7-1 & \end{array}$$

That is, the division of $f(x)$ by $x-1$ gives a quotient of x^2-8x-7 and a remainder of -1. Call this remainder r_0. Let us now divide the quotient by $x-1$. We have

$$\begin{array}{r|l} 1-8-7 & 1 \\ 1-7 & \\ \hline 1-7-14 & \end{array}$$

That is, division of our first quotient x^2-8x-7 by $x-1$ now gives the quotient $x-7$ and remainder -14. Call this remainder R_0. We now note that

$$h_0=-\frac{r_0}{R_0}.$$

Translate the above into general notation. State and prove a general theorem. Do you know what R_0 is? In the calculus, one speaks of the slope of a line tangent to a curve. The slope of the line tangent to the curve $y=f(x)$ at the point (x_0, y_0) is often designated $f'(x_0)$. This is the derivative of $f(x)$ with respect to x evaluated at $x=x_0$. Now

$$R_0=f'(x_0).$$

Alas, we have reached the calculus. For our particular problem

$$R_0=f'(x_0)=3x_0^2-9\cdot 2x_0+1$$
$$=3x_0^2-18x_0+1$$

and

$$f'(1)=-14.$$

We illustrate computation of the derivative using the function

$$g(x)=x^2.$$

Now

$$g(x+h)=(x+h)^2=x^2+2hx+h^2.$$

Thus,

$$\frac{g(x+h)-g(x)}{h}=2x+h.$$

By definition

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=2x.$$

Can you generalize?

To summarize, if x_0 is an approximation to a root of $f(x)=0$, then a better approximation is usually obtained by computing

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Is the latter formula useful if $f(x)$ is not a polynomial?

At the start of writing this article, I had no intention of ascending to the calculus. But mathematical sweetness got the best of me, and I hope it does the same for you.

In numerous texts on algebra and theory of equations, one finds a discussion of a method for finding real roots of a polynomial. The technique, based on synthetic division, is called Horner's method. It requires continuation of the synthetic division process beyond that required for the Newton-Raphson process. Thus, more work is involved. Further, the convergence in Horner's method is slower than that of the Newton-Raphson technique. Conclusion: *Horner's method should be relegated to the class of useless objects.* (Textbook writers, teachers, and students please take note.)

8. How can you use the ideas developed here to solve for the unknowns x and y, given the two equations $f(x, y)=0$ and $g(x, y)=0$? For example, find the real solution x and y given that $4x^3-27xy^2+25=0$ and $4x^2y-3y^3-1=0$. Hint: $x=1$, $y=1$ is an approximate solution. Let $x=1+h$, $y=1+k$. Substitute these in the equations whose solution we seek and after neglecting certain terms derive two linear equa-

tions involving h and k. Solve for h and k and repeat the process to find improved approximations. The coefficients of h and k are known as partial derivatives. If you know something of the calculus, more difficult equations can be handled. How do you apply the above idea to find the real values of x, y, and z given the nonlinear equations:

$$x^2+2xyz-z^2y^2=8;$$

$$x+y-3z=-7;$$

$$x^2+y^2-z^2=0?$$

Hint: $x=2$, $y=1$ and $z=3$ is an approximate solution. Put $x=2+h$, $y=1+k$, $z=3+m$, and proceed as before. Is further generalization possible? It should be noted that using the ideas presented, solution of r nonlinear equations in r unknowns requires the solution of r linear equations in r unknowns. This is one of many reasons why solution of linear equation systems is so important in modern mathematics. We hope to deal with this topic in a future article.

In conclusion, we present some references for further study.

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Mathematical Profiles

Several years ago an ad hoc committee of the Division of Mathematics of the National Research Council made a survey of training and research in applied mathematics in the United States. One of the recommendations of this committee, made as a result of the survey, was that the Division establish a standing Committee on Applications of Mathematics with the following functions:

(a) To facilitate cooperation among organizations concerned with various aspects of mathematics in applied settings.

(b) To call attention to the emergence of new areas in which significant applications of mathematics may be possible.

(c) To serve as a focus for the continuing scrutiny of problems concerned with training and research in mathematics as related to its applications.

(d) To take whatever steps are deemed appropriate to enhance the effectiveness of mathematics in its applications.

Accordingly, in October, 1954, the Chairman of the Division appointed a committee of eight members, with Dean Mina S. Rees, of Hunter College, as chairman. Dean Rees continued as chairman until the summer of 1956, at which time she asked to be relieved of the chairmanship, and the present writer was appointed in her place. In accordance with the usual policy of the Council, members are regularly appointed for terms of three years with two or three replacements each year.

The Committee met, soon after it was formed, to determine a course of action, and for reasons that hardly need elaboration here, its attention quickly centered on problems of training. Its major concern has been with the growing demands for mathematicians in government and industry, and the insufficient numbers of students being prepared to meet these demands. This problem, it seemed, must be attacked at the high school level, and Dean Rees proposed a project which is now well under way and will perhaps be of interest to others.

While doubtless many factors contribute in tending to repel even superior students from courses in mathematics, one of the important factors is lack of information about the careers that are open to mathematicians, and a scant and often distorted conception of mathematicians and their activities. The committee felt

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General mathematics for college freshmen[1]

KATHRINE C. MIRES, *Northwestern State College, Alva, Oklahoma.*

A study of students' functional competency—and a proposal.

IN ANY DISCUSSION of general education the role of mathematics is likely to be considered. There is, however, a wide range of opinion as to just what is the place of mathematics in general education, especially at the college level. Those who are convinced of the need for including mathematics in general education programs for colleges are faced with the difficult question of what should be the nature of such mathematics work. The study reported here was concerned with one type of course in mathematics for general education at the college level.

STATEMENT OF THE PROBLEM

The problem was that of determining the extent to which there is need for a college-level course in mathematics for general education that has as one of its chief purposes the improvement of functional competence in mathematics and the investigation of the nature of the content related to functional competence that should be included in such a course. As used in this study the term "functional competence in mathematics" means an understanding of and an ability to use the basic mathematical concepts included in the Check List[2] of essentials for functional competence in mathematics which was prepared by the Commission on Post-War Plans of the National Council of Teachers of Mathematics.

Specifically, answers to the following questions were sought:

1. To what extent do freshmen entering selected four-year colleges in Oklahoma have an understanding of and an ability to use the essentials for functional competence in mathematics that were recommended as a part of the general education of all citizens by the Commission on Post-War Plans of the National Council of Teachers of Mathematics?

2. What are the implications of the results obtained in answering the first question as to (a) how great the need is for a college-level general mathematics course that includes work designed to improve functional competence and (b) what mathematical concepts involved in the essentials for functional competence should be included in such a course in mathematics for general education?

PROCEDURE

In order to find answers to the above questions, the writer had the *Davis Test of Functional Competence in Mathematics*[3] administered to 1811 entering college freshmen at the six state colleges in Oklahoma in the fall of 1955 and studied the results. The Davis Test was selected because it was constructed to test those

[1] This article is a summary of the author's Ed.D. dissertation, "The Need for and Nature of One Type of Course in Mathematics for General Education at the College Level," University of Oklahoma, 1956.

[2] "Guidance Report of the Commission on Post-War Plans," THE MATHEMATICS TEACHER, XL (November, 1947), 318–19.

[3] David John Davis, *Davis Test of Functional Competence in Mathematics* (Yonkers-on-Hudson, New York: World Book Company, 1951).

concepts covered in the above-mentioned Check List. Although the test was designed for use with grades nine through twelve, it was considered satisfactory to use it with the college students since the group tested were entering freshmen. The six state colleges in Oklahoma are Central State College, Edmond; East Central State College, Ada; Northeastern State College, Tahlequah; Northwestern State College, Alva; Southeastern State College, Durant; and Southwestern State College, Weatherford. These institutions were selected because they are similar in size and function and draw students from the entire state.

Although the colleges were requested to administer the test to all freshmen entering the institutions for the first time in the fall of 1955, it proved to be impossible to test all entering freshmen. The 1811 students tested represented 80.2 per cent of all entering freshmen at the six institutions. The lowest per cent of entering freshmen included in the group tested at any college was 67.7 and at only one other institution was the per cent less than 80. Because a large percentage of the total group of entering freshmen took the test and because those taking the test were not selected in any way, the subjects were treated as a total population in dealing with the data.

The test results were studied by means of a statistical treatment of scores and an item analysis, in which the number and per cent of the subjects who answered each question correctly were determined. The items were then divided into seventeen groups, each group containing items that dealt with the same basic mathematical concept. Although some of the items required knowledge of more than one of the concepts covered in the test, each item was placed in the group that, in the opinion of the writer, represented the central concept tested by the item. Each question was placed in only one group. Then the mean percentage of the subjects giving a correct response to the items in each group was computed and used as an indication of the level of competence of the subjects on the concept tested by the items in that group.

An additional means of interpreting the test results was obtained by considering the opinions of a group of twenty-six selected mathematics teachers and administrators as to which items on the Davis Test deal with knowledge that is essential for all college students. All of the judges were staff members of either public schools or colleges in Oklahoma. The group consisted of eight public school administrators, five college deans, and thirteen mathematics teachers. No attempt was made to select a random sample of mathematics teachers and administrators because the nature of the task the judges were to be asked to do made it necessary to have individuals who had the background and interest required for a difficult and time-consuming piece of work.

Each judge was asked to indicate independently which items on the Davis Test he believed were essential as a part of the knowledge of the college student taking work in a variety of general education fields and in a major field not requiring extensive training in mathematics. By determining the total number of items marked essential by each judge and finding the mean of these totals it was possible to arrive at some indication of a minimum satisfactory score on the test in the opinion of the judges.

Results

In addition to the total score, which was the total number of items answered correctly, two part scores were determined for each subject. These part scores were the number of odd items answered correctly and the number of even items to which a correct response was given. From these part scores the corrected split-half reliability coefficient of the test for the group of subjects was found to be .90.

Since the score made by an individual on the Davis Test was the number of

items answered correctly, the possible score was eighty. The range of scores made by the subjects in this study was from four to seventy-eight. A frequency distribution of the scores made by the 1811 entering college freshmen is presented in Table 1. The mean of the scores was 30.8, and the standard deviation was 11.7. The median was 29.2. Fifty per cent of the group made scores between 22.3 (Q_1) and 37.9 (Q_3).

TABLE 1

FREQUENCY DISTRIBUTION OF SCORES FOR 1811 ENTERING COLLEGE FRESHMEN ON THE DAVIS TEST

Class Interval	Frequency
75–79	1
70–74	0
65–69	9
60–64	19
55–59	42
50–54	58
45–49	115
40–44	140
35–39	214
30–34	289
25–29	345
20–24	291
15–19	196
10–14	63
5– 9	27
0– 4	2

Since it was assumed in this study that the concepts and abilities included on the Check List were valid essentials for functional competence in mathematics and that the performance of a college freshman on the Davis Test revealed his functional competence in the mathematics covered by the items on the Check List, only those students who answered correctly all or nearly all of the eighty items on the test can be considered to have attained a desirable level of functional competence in mathematics. From the data presented above it is evident that the achievement of most of the freshmen was far below this desirable level.

When the scores of the subjects were compared with the end-of-year percentile norms for high school seniors, it was found that the performance of the college freshmen was slightly poorer than that of the high school seniors. However, the data indicated that the performance of both groups was relatively poor in view of the fact that the test was designed to measure concepts and abilities considered necessary for all citizens by a national committee of mathematics teachers.

The minimum satisfactory score obtained by finding the mean number of test items considered essential by a group of twenty-six selected mathematics teachers and administrators was sixty-two, which is in sharp contrast with both the mean (30.8) and the median (29.2) for the group of college freshmen. Even Q_3 (37.9) for the group of subjects was twenty-four points below the minimum satisfactory score. In fact, only twenty-one or 1.16 per cent of the 1811 college freshmen made a score equal to or greater than the minimum satisfactory score as determined by the judges. Furthermore, the number of items considered essential by twenty-five of the twenty-six judges was greater than the third quartile for the group of college freshmen.

The item analysis phase of the study also revealed the widespread lack of functional competence among the subjects. Only twenty-three of the eighty test items were answered correctly by more than 50 per cent of the college freshmen, and only four questions received correct responses from more than 75 per cent of the group. No question was answered correctly by as many as 85 per cent of the subjects.

In the phase of the item analysis in which the test questions were divided into seventeen groups each of which contained items that dealt with the same basic mathematical concept, it was found that the mean percentage of the freshmen giving correct responses to the items in each group varied from 73.7 to 4.2. However, in only four groups did the mean percentage of students answering correctly exceed 50, and in only one of these

four groups was the percentage greater than 60.

Among the thirteen groups of items for which the mean per cent of the freshmen giving the correct response was less than 50, there were three that contained only one item each. These three groups were discarded. The titles of the ten remaining groups were arranged in order according to the mean per cent of the subjects giving the correct response, with the group having the lowest per cent first. The result was this list of topics on which the 1811 freshmen showed the least functional competence.

1. Drawing conclusions
2. Estimating answers
3. Measurement
4. Use of approximate numbers
5. Basic geometric concepts
6. Reading and interpreting tables
7. Use of formulas
8. Consumer problems involving per cents but also requiring some other knowledge
9. Basic algebraic simplification
10. Ratio and proportion

Conclusions and Recommendations

The conclusions drawn from the study may be summarized as follows:

1. Only a very few of the 1811 entering college freshmen tested at the six state colleges in Oklahoma in the fall of 1955 had a satisfactory understanding of and an ability to use the essentials for functional competence in mathematics that were recommended as a part of the general education of all citizens by the Commission on Post-War Plans of the National Council of Teachers of Mathematics.

2. There was a definite need in the six state colleges in Oklahoma in the fall of 1955 for a general mathematics course which included work designed to improve functional competence.

3. The group of entering freshmen tested lacked functional competence in all areas covered by the test, but they showed greater deficiency in some phases than in others. Greatest lack of competence was shown on the ten topics listed above.

4. A course in mathematics for general education that has as one of its chief purposes the improvement of functional competence might well include work on any of the mathematical concepts covered by the Davis Test, but emphasis should be placed upon the ten topics on which greatest lack of competence was shown.

The recommendations are concerned with providing and developing a college-level course in mathematics for general education. It is recommended that each of the faculties of the six state colleges in Oklahoma give consideration to including in the requirements in the area of general education a mathematics course that includes work designed to improve functional competence.

It is important that the course be organized and presented in a manner that emphasizes the development of understanding of mathematical concepts rather than the improvement of mathematical skills and the memorization of facts. The conclusion that emphasis should be placed on the ten topics on which least competence was shown should not be interpreted as meaning that each of them should be taken up as a separate unit of the course and taught without showing its relationship to other parts of the work. It is recommended that these and other topics to be considered be organized about broad mathematical concepts as a means of unifying and making meaningful the various specific topics. For example, the idea of functional dependence can be used as a unifying concept through which to teach the following of the ten topics: reading and interpreting tables, use of formulas, ratio and proportion, and some aspects of drawing conclusions and of basic geometric concepts. Another area covered in the Davis Test that can also be studied best in connection with functional dependence is reading and interpreting graphs. Other possible broad mathematical concepts for use as a means of unifying the various parts of a course of the type under consideration will occur at once to those interested in developing such material.

• MATHEMATICS IN THE JUNIOR HIGH SCHOOL

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Mathematical terms in everyday expressions

by John P. McIntyre, Florida Department of Education, Tallahassee, Florida

The mathematics teachers in Kinloch Park Junior High School wish to share with other mathematics teachers an activity that they and their pupils particularly enjoyed. It all started quite casually one lunch period when one tired-out math teacher sighed that she had been going around in circles all morning. Another teacher hastened to add that teaching was a vicious circle anyway. And then when a third remarked about the human equation in teaching, particularly in mathematics, they decided to go off on a tangent and collect terms from mathematics used in everyday expressions.

The mathematically-minded teachers realized that they were stretching a point to include some of the terms below as strictly mathematical. They agreed, however, that it would be well to add one or more for good measure, rather than to subtract.

Teachers report that pupils enjoyed completion tests based on this information. Symbols placed alongside blanks to be completed aided in the recollection of correct terms. In this manner, the tests incorporated spelling and vocabulary as well as a knowledge of mathematical terms.

fraction of a second
cube steaks
square dance
volume of business
sphere of influence
circle of friends
prime ribs of beef
acute appendicitis
equal rights
square knot
zero hour
hot rod
double date
pizza pie
Max Factor makeup
times are tough
"Tea for Two"
wage scale
"he's a square"
eternal triangle
ice cream cone
master's degree
100 per cent right
batting average
the Great Divide
every inch a king
ceiling zero
exponent of justice
second degree murder
the great unknown
out of all proportion
"Lost Chord"
welcome sign
round steaks
divide and conquer
line of least resistance
point of order
square deal
rule the roost
straight and narrow path
negative attitude
Pentagon Building
sum and substance
"X marks the spot"
capacity audience
telegraph
set an example
cheaper by the dozen
plus fours
"Crossing the Bar"
Joan of Arc
rule of thumb
vicious circle
baby formula
find all the angles
intelligence quotient
the Golden Rule
hard times
octagon soap
split second
safety in numbers
literal translation
first signs of spring
constant as the North Star
take an interest in
integral part

A column of unofficial comment

Thoughts on teacher training

by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

Teaching consists of behavior intended to result in students acquiring knowledge. It is a rational activity, that is, it is based on voluntary choices regarding selection of subject matter and methods of teaching. These choices are made explicitly or implicitly in terms of certain principles, and can be justified by the teacher if they are called into question. The justification proceeds, as does every case of justification, by arguing that the choices are consistent with a set of criteria (principles). This would seem to imply that teachers need to learn certain principles relevant to the process of acquiring knowledge in addition to the knowledge itself which is to be taught the students. This, briefly, is the argument for courses in methods of teaching and once was the argument for courses in principles of elementary or secondary education.

We have done a respectable job of teaching teachers knowledge about students, e.g., how they mature physically, intellectually, emotionally, and socially; how they are motivated; how they learn; and what they do when they are frustrated. And, in light of the exigencies in the public school that usually require a teacher to teach in different subject-matter fields, we have done a respectable job of teaching teachers knowledge about the subjects they are to teach. But we have done less than a respectable job of teaching teachers about knowledge per se and about the linguistic processes which they must carry on in helping students acquire knowledge. Since space is limited, I shall only consider some of the linguistic processes.

Every teacher interprets the meaning of symbols. Sometimes this consists of interpreting a single symbol such as a word or a mathematical symbol like $f(x)$. Sometimes it consists in interpreting the meaning of an entire sentence. Interpretation is done in either of two ways. The teacher may state an expression that purports to be synonymous with the word or sentence that is being interpreted. Or, if he is interpreting an empirical statement, he may indicate the truth conditions of the statement without presenting an implied synonymous statement. To interpret he may have to define, in which case he must choose what kind of a definition to employ: ostensive, denotative but not ostensive, analytic, or operational. Each of these kinds of definitions has advantages and each has disadvantages. It might be expected that a teacher who is aware of these kinds of definitions and knows the advantages and disadvantages of each kind would be more effective in carrying out the process of interpretation than would one who does not have this knowledge.

Every teacher uses abstractions. He speaks, for example, of *the* linear function apart from particular functions. He helps students abstract. Once he can get them to recognize the analogy between two or more things, that is, recognize different media for the same form, it is but a short step to abstraction, that is, recognition of

the common form regardless of the medium.

Every teacher makes inferences. Some of these are valid, that is, in consonance with laws of logic. Some are invalid—not in consonance with laws of logic. Some are necessary inferences, that is, inferences in which it is impossible for the conclusion to be false and the premises true. Some are probable—supported by partial rather than complete evidence—or, to put it another way, inferences in which it is possible for the conclusion to be false even though all the premises are true.

Every teacher discriminates between true propositions, false propositions, and propositions which are neither true nor false. Some statements that a teacher uses are statements about how a particular language is to be used. Such statements are necessarily true, that is, there is no conceivable empirical evidence that will falsify such statements. Examples of this kind of statement are stipulated definitions and logical formulae. Many of the propositions of mathematics are necessarily-true statements.

Some statements that a teacher uses in class are statements about "reality." An example of this kind of a statement is "Chicago is the largest city in population in Illinois." Such statements, called synthetic statements by philosophers, are verifiable empirically. They attain a truth value by observation and are in the form of probable inferences.

Some propositions, particularly in mathematics, are neither true nor false. For example, the equation $2x+6=12$ is neither true nor false. It becomes true or false after the variable is replaced by a numeral. Another kind of statement that is neither true nor false is a prescription. Prescriptions are commands, or, in softened form, exhortations. Teachers usually use prescriptions like "check your answers," "subtract 6 from each side of the equation," because they are more effective in determining behavior than are generalizations.

Teachers often utter and respond to sentences containing a value word like "good," "elegant," "neat," and others. Such sentences are anomalous as far as truth-value is concerned. They are not entirely descriptive; they are used to rate something. "x is good," depending on the context, may mean "x has the properties a, b, and c," in which case "good" is synonymous with "having properties a, b, and c." "x is good" may also be used to mean "I approve of x," in which case it is an empirical statement although not verifiable in a public sense. Finally, "x is good" may be used in the sense of a prescription, "do x," "approve of x," or "choose x," as the case may be. In this case what has been said about a prescription applies to a value sentence.

Every teacher explains and has his students explain, that is, answer the questions "why?" "how do you know?" or "why do you believe this?" Some explanations are complete, giving both the generalization and the proposition, subsuming what is explained under the generalization. Most are incomplete, giving only the generalization or the statement of implied relevance, but not both. Some explanations are true; others are false, and still others we do not feel can be assigned a truth-value in terms of our present knowledge.

Every teacher justifies and asks his students to justify beliefs, values, decisions, plans, and actions—that is, show that the beliefs, values, decisions, plans, or actions are consistent with certain acceptable criteria. Sometimes these justifications satisfy all the conditions of a proof but are not convincing. And sometimes they neither satisfy the conditions of a proof nor are convincing.

To be sure, a teacher will carry on these linguistic activities without ever having been trained in them. But, if we believe, as we seem to believe, that knowledge affects behavior, knowledge of these activities should improve the teacher's effectiveness. Where does the teacher acquire

such knowledge? Not in the usual undergraduate education courses. If the course is billed as "Principles of Secondary (or Elementary) Education," it tries to present a survey of such topics as the historical development of the public schools, their organization and administration, the purposes of education, the curriculum, the extra-curricular activities program, the guidance program, the school and the community, and the professional preparation of teachers—no one of which is treated thoroughly.

If the course is billed as a methods course, usually it is overly oriented to psychological theory. Please note that I said *overly* oriented. I do not want to give the impression that psychological theory is unimportant. Each of the linguistic processes mentioned above has a psychological aspect, and each has a logical aspect. Both psychology and logic are important. It is the latter that I am arguing is being neglected in education courses in methods of teaching and principles of elementary or secondary education. As a result, many teachers do not know the logic of the processes mentioned above. And, consequently, they are at a loss as to what to say when a student questions whether a particular assumption is necessary, an inference valid, a statement true, an explanation correct, or a justification adequate. They have no criteria to fall back on, and, if they are not careful, resort to authoritative opinion—"it's so because I say so," or "the textbook says so."

How shall prospective teachers be provided the training alluded to? Partly by a course in logic including semantics. Partly by changing courses in principles of secondary (or elementary) education from courses which try to cover the waterfront by providing a "quickie" of each of many complex problems to courses which focus on the linguistic processes mentioned above and how they can be carried on effectively in the classroom.

We cannot leave such instruction entirely to special methods courses even if textbooks for these courses were to devote attention to this kind of instruction. There are too many problems which, though related to the language used in the classroom, are peculiar to the subject field and must be considered in special methods courses. We might assume that if a "principles" course, treating what has been indicated, precedes a special methods course the latter course itself would be more effective. And this, in turn, should make student teaching prior to employment and teaching following employment more effective.

The trouble does not end when we get science to do its new jobs; more likely, it only begins. When science delivers the goods now ready or on order, not a single additional cubit of wisdom will thereby be brought into the world. For there still remains the biggest problem of modern man—perhaps even bigger than war: what to do with himself. As he ceases to be a creature of endless toil, poverty, and famine, he is apt to find himself liberated into nothingness. His leisure time can become more of a curse than the plagues of old.—*Taken from an editorial in* The Saturday Review, *October* 27, 1956.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

A Short Dictionary of Mathematics, C. H. McDowell (New York: Philosophical Library, Inc., 1957). Cloth, xiii+63 pp., $2.75.

This book, unintentionally, is a noteworthy step towards a list of language patterns which, as a duty, mathematics teachers should *avoid*. The jacket flap promises: "It will be particularly welcomed by those who have always felt handicapped because they were 'no good at math' in school." Certainly, the mathematical world will not welcome it. For those who were "no good at math" it is probably an effective display of the reasons why people who could have been "good at math" were not.

One hopes to be able to laugh at this entire work as a prank of a clever mathematician. In fact, with a little more polish, it could have been an excellent satire on the mish-mash of words now used in teaching precollege mathematics. If it were viewed as such a satire, much progress could come from shunning the definitions given in it; otherwise, it is an unfortunate move in the wrong direction.

Here are six examples taken from the book:

1. "REAL NUMBERS are actual numbers as opposed to imaginary or impossible numbers. As 3, 7, 25, etc."

To illustrate how easily this could have been converted into a vivid satire, how about: Real numbers are those numbers that are really real?

Many people have already commented on the necessity of showing students that complex numbers are no less "real" than other kinds of numbers.

2. "COMBINATIONS are the number of different groups which can be made of any number of individual numbers taken in sets of a given number."

This is quoted merely as an example of unintelligibility.

3. "LAW (Distributive) [sic] means that addition and subtraction may be performed in any order."

This is false. Interpreted charitably, the author must be thinking about commutative laws. But then one infers that the operation of subtraction is commutative, which is also false. Of course, subtraction *can* be performed in either order (with different differences), but it is doubtful the author was considering such subtleties.

4. "ALGEBRA is that part of mathematical science in which letters or characters or symbols *instead of numbers* are used for the calculations." (Italics mine.)

Here is support for the view too often held by students that algebra says good-bye to numbers. Students should learn instead that in (elementary) algebra letters are used to talk about *numbers* in order to say a lot, briefly.

5. "ZERO is a figure, character or symbol (written 0) and has the same meaning as cipher, naught, nought, or nothing."

Should not a student after reading this be tempted erroneously to say: The equation $3x+5=5$ has the root zero; but zero is nothing, so the equation has nothing for a root; *i.e.*, the equation has no root?

6. "COMPOSITE NUMBER is a number which is the product of any two or more numbers . . . ."

From this definition we deduce that every number is a composite number because every number is, for example, the product of half of itself by two. If every number is a composite one, why introduce the term? Of course, the charitable interpretation is that the author *meant* (correctly) to say that a composite number is a natural number which is not a prime number and not 1. The student who was "no good at math" will hardly guess this.

This list of errors and possible reader inferences could be extended manifold. It was difficult to choose only six among so many gems.

This book has a second part dealing with geometry and trigonometry which is a routine list of "facts" from these subjects. The errors are less glaring here, but there is nothing new or exciting in it.

If this "dictionary" were to be taken seriously, there would be many serious omissions to list. However, taken as it is, each omission is another word which has escaped unsullied.—*D. A. Page, University of Illinois, Urbana, Illinois.*

Introductory College Mathematics, Robert W. Wagner (New York: McGraw-Hill Book Co., Inc., 1957). Cloth, xiv+430 pp., $5.00.

The purpose of this text in the words of the author is "to provide a basis for an introductory course in college mathematics which leaves the student with a concept somewhat closer to a mathematician's view than is usually attained." Its design gives the student an introduction to algebra, trigonometry, analytical geometry, and the calculus. The final two chapters deal with "Interest and Annuities," and "Statistics

and Probability." The appendix contains tables to be used with the text.

The material of the book is organized with a view to the three-period-per-week course meeting over a year's time. In the preface the author states that the book contains "more material than is normally covered in a year" on such a schedule. This means that in the ordinary situation the instructor must either omit certain parts of the material or cover it all in a superficial manner. Either choice might leave the first-year student somewhat confused.

The avowed goal of the treatment of the material is "to achieve an understanding with a minimum of vocabulary and formulae memorization." This we believe is accomplished. At the same time there seems to be a wealth of questions to provoke thought and sufficient exercises to use for drill work.

In Chapter Two there is a rather unique parallel presentation of equations and inequalities that should leave the student with a good grasp of inequalities, but may at the same time place too much emphasis on them as compared to the rest of the material of the text. (Chapter One deals with "Numbers and Their Uses.") After Chapter Two, the function concept is introduced and serves to unify the remainder of the material on algebraic and non-algebraic classes. The work on progressions is split up into two sections. The arithmetic progressions are taken up in Chapter Four along with linear functions, while geometric progressions are introduced in the next chapter with exponential functions. In Chapter Seven the trigonometric functions are introduced and defined by use of the unit circle. Angle is defined as an amount of rotation. The symbol $\sin^{-1}(x)$ is defined as an angle in degrees while arcsin (x) is used to represent a number or radian measure.

The text is well written with very few errors. One that should be mentioned occurs in the introductory statements to the student on page *x*. The text gives as one of the basic propositions from geometry, "Two triangles are similar if, and only if, the ratios of the sides opposite equal angles are equal. This is often phrased as 'the sides of similar triangles are proportional.' " The two statements should not be classified as being the same since one is the converse of the other.

In our observation this type of text often runs to one of two patterns. It is either a stereotyped discussion of the standard topics from algebra through the calculus, or it tends to have the tone of "fun with functions" approach. The present volume steers a wise course between these two extremes. If any criticism be made at this point it is that the author tends to be on the "sober" side. However, this is meant as a comment only and it is not meant to detract from what we believe to be a good work. As a text for a first year course in introductory college mathematics it should certainly prove satisfactory.—*Paul F. Iverson and Allan H. Paine, Potomac State College of West Virginia University, Keyser, West Virginia.*

Mathematics of Finance, Hugh E. Stelson. New York, D. Van Nostrand Company, Inc., 1957. Cloth, xii+327 pp., $5.50.

This book is designed for use in a college course taken by students who are interested in obtaining a major in mathematics or business administration. While a review of the algebra needed for the course is included in the book, the student should have had a course in college algebra before studying *Mathematics of Finance*. In order to cover the topics thoroughly, a two-semester course would be required.

One of the outstanding features of this book is the attempt to bring into the work many problems which are likely to be met by the students in life situations. The author also tries to define and explain many of the terms and procedures used in the business world. At times he becomes so engrossed in doing this that the explanations of the mathematical processes to be used are neglected. It is in these cases that it seems as if the book is written primarily for business administration students.

Many of the explanations of specific topics are very brief and would need to be supplemented extensively by the teacher. The illustrative problems included are for the most part quite good. In my opinion more exercises should have been included. Since in most cases there are only a few following the explanation of a particular topic, the teacher is not given much of an opportunity for selection. A very good feature is that a list of problems covering the entire chapter is included at the end of many of the chapters.

As with most textbooks, explanations of topics precede the citing of a problem which would require that material for solution. The motivation for learning the material and the understanding of it would be increased if the order of presentation were reversed.

The general case of the annuity is treated in a separate chapter after the ordinary annuity has been thoroughly treated. Only a brief section is included concerning the equation of value. Since this is a very important topic, more stress should be placed on it.

The notation used seems to be fairly consistent with that used in actual business practices. Interest rates used are in line with those used in the business world at the present time.

For the most part this book is at least as good as most of the mathematics of finance books. If it has a distinctive feature, it is that a large part of each explanation is devoted to a discussion of actual business practices.—*Harold W. Brockman, Capital University, Columbus, Ohio.*

The Mathematics of Investment, Roger Osborn. New York, Harper and Brothers, 1957. Cloth, viii+279 pp., $4.25.

This book is designed for use in a three-hour, one-semester college course. Before studying this book, the student should have had a course in algebra which included work with geometric progressions.

The material included in the book is very similar to that included in most mathematics of investment books, with the exception that no material concerning life insurance is included. The author considers three cases when discussing annuities. These three cases are: when the payment intervals and interest conversion periods coincide; when there are k conversions of interest during each payment interval; when there are p payments in each conversion period. Most books do not make as much of a distinction between the last two types as does this book.

The explanations given are quite complete and detailed. The author attempts to familiarize the student with much of the terminology used in the business world and tries to help him to understand the background of all operations performed. The illustrative problems are well selected and well explained. A student should have no difficulty following the explanations. There is an ample number of exercises included and these are realistic problems with realistic interest rates. At the end of the book there is a supplementary list of problems covering all of the work.

Only a rather short section treats the equation of value. Since this is a concept which is fundamental to all of the work, more emphasis should be placed upon it. This concept should be stressed throughout all the work and is not in this book.

The notation used in the book is only fair and becomes somewhat unrealistic at times. Letters are sometimes used which are quite unusual in connection with the topic under consideration.

In my opinion, this book follows rather closely the usual pattern of presenting the material covered and does not have anything to offer which would cause it to be any better than other books concerning mathematics of investment.—*Harold W. Brockman, Capital University, Columbus, Ohio.*

Trigonometry for Secondary Schools, Charles H. Butler and F. Lynwood Wren. Boston, D. C. Heath & Company, 1957. Cloth, vii+360 pp., $2.96.

This trigonometry book, written by two renowned educators, reflects many sound principles of pedagogy. The purposes of the chapters are made clear to the students by short introductions. There are attempts to promote motivation through realistic problems. The development is slow and gradual, and there appears to be an adequate amount of drill and review material covering both present and past learning (e.g., exponents). The authors employ mnemonic devices, such as the "function hexagon."

The form of solution of illustrative problems is neat and clear, and the authors are meticulously careful to explain and use correctly approximate numbers. Scientific notation is used to good advantage.

If the teacher requires a very traditional textbook in trigonometry, this book will be worth his consideration. It starts with the usual discussion of similar right triangles and works into the ratios in what the reviewer would call the old-fashioned way. An idea of the plan of the book may be secured from the division of space: 103 pages on "right-triangle trigonometry," 106 pages on "general trigonometry," and 38 pages on spherical trigonometry. It is likely that many teachers would start a trigonometry course on page 104, referring from time to time to review sections in the first part. It is unlikely that the section on spherical trigonometry would be meaningful to a student without a course in solid geometry.

Analytic trigonometry, the "new" trigonometry, is not emphasized in this book: for example, the function concept; trigonometric functions of numbers; identities; trigonometric equations; and so forth.

There are a few specific criticisms which may be useful to the teacher considering this textbook:

1. "The value of the tangent ratio of an angle depends upon the size of the angle" (p. 11). In context, since the student applies the statement only to the right triangle, it may be inferred that as θ increases, tan θ always increases, and vice versa.
2. There is an unfortunate implication that $0.333\dot{3}\ldots$ is an approximate number (p. 36).
3. "The precision of a recorded measurement is determined by the size of the absolute error" (p. 38). This does not *define* precision adequately. Furthermore, since the size of the absolute error is never obtainable, a definition in these terms would not be useful.
4. The discussion of "rounding an integer" certainly seems inappropriate, since an integer is an exact number (p. 42).
5. "Obviously, a value for the area given to ten-thousandths of a square foot cannot be justified when the measurements of length and width are each given to hundredths of a foot" (p. 43). The reviewer feels that the word "obviously" is gainsaid by billions of students.
6. Most books have problems with absolutely vertical cliffs, so that neat right triangles can be drawn. The reviewer takes this opportunity to inquire as to the location of such cliffs (p. 47).
7. The answer "$h=48.74$ feet or 49 feet" appears to indicate that either answer is acceptable (p. 84).
8. The reviewer feels that the publisher could have done much more with the book in terms of format, print, size and nature of diagrams, and the addition of color. The diagrams are small and poorly lettered, and pages are insufficiently "broken up." This makes the book start with two strikes against it through no fault of the authors (who herewith have my sympathy).—*Irving Allen Dodes, Bronx High School of Science, New York, New York.*

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia, and Joseph N. Payne, University of Michigan, Ann Arbor, Michigan

Fun with graphs

by Paul S. Jorgensen, Carleton College, Northfield, Minnesota

Here is a way to brighten up your practice exercises or your tests on graphing. The object is to plan a series of graphing exercises which will result in a picture. The first set of exercises below gives practice in plotting points and interpreting the slope of a line, as well as graphing linear equations, inequalities, and circles. The overall result, as you can see, is a Christmas tree (complete with ornaments), and the greeting, MERRY XMAS.

The author started using this approach in an effort to find some way to keep students interested and working on the day before Christmas vacation. A test was assigned for the day, in itself enough to convince the students that the teacher was, indeed, Scrooge in disguise. Needless to say, the test was started in an atmosphere of grumbling and malcontent. As the students neared completion of the test the "climate" changed to one more in keeping with the approaching holiday. Of course, some of the students who were ill prepared found no holiday spirit in the test, since their results were far from satisfactory.

The test can be completed by most students in a fifty-minute class period if the teacher prepares the graph paper in advance so that the student does not need to spend time arranging axes in the proper places on the page and deciding on the proper units. All that is necessary is an indication of position of the origin and one or two numbers to fix the units along the axes.

Graphing test

1. From each point $(-7, 15)$, $(2, 15)$, $(0, -4)$, $(-2, -4)$, $(-5, 15)$, $(-4, 15)$, $(-1, 15)$ draw line segments with infinite slope, that is, the slope is not defined, and with length that increases the ordinate by 3 units.

2. With the following points as centers, draw circles with a radius of 2/5 of one unit: $(-1, 7)$, $(2, 5)$, $(1, 10)$.

3. Starting at $(-1, 11)$ draw a line segment with a slope of $+3/2$ and with a length that decreases the ordinate by 3 units.

4. Join $(3, 5)$ and $(5, 2)$ with a straight line segment.

5. From each point $(-4, 18)$, $(4, -4)$, $(2, 18)$, $(4, -1)$, $(-1, 16\frac{1}{2})$, $(-4, 15)$, $(-1, 18)$, $(2, 16\frac{1}{2})$, $(4, -2\frac{2}{3})$, draw line segments with slope zero and a length that increases the abscissa by 2 units.

6. From each point $(-5, -1)$, $(2, 8)$, draw line segments with a slope of $-\frac{3}{2}$ and a length that decreases the ordinate by 3 units.

7. Join the following pairs of points with straight line segments: $(6, 16)$ and $(6, 15)$; $(1\frac{2}{3}, -3)$ and $(2\frac{1}{3}, -3)$; $(0, 2)$ and $(-5, 2)$.

8. Draw a line segment with a slope of $-\frac{3}{2}$ from the point $(2, 11)$ to its y-intercept.

9. From each point $(6, 16)$, $(-1, -3)$, $(-6, 16)$, draw line segments with a slope of $+2$ and a length that increases the ordinate by 2 units.

10. From each point (0, 14), (−2, 8) (−3, 5), (−3, −1), draw line segments with a slope of $+\frac{2}{3}$ and a length that decreases the ordinate by 3 units.

11. Graph

$$y=-\frac{3}{2}x+\frac{25}{2}$$

for $1\leq x\leq 3$.

12. Join the following pairs of points with straight line segments: (2, −1) and (1, −4); (1, 18) and (1, $16\frac{1}{2}$); (4, 18) and (4, $16\frac{1}{2}$).

13. From each point (−7, 18), (5, 18), (−2, −1), draw line segments with a slope of −2 and a length that decreases the ordinate by 2 units.

14. Graph $y=-3x+5$ for $2\leq x\leq 3$.

15. Plot all points (x, y) such that $0\leq x\leq 5$, $y=2$.

16. Plot all points (x, y) such that $x=\frac{2}{5}$, $0\leq y\leq 2$.

17. Join the following pairs of points with straight line segments: (−4, $16\frac{1}{2}$) and (−3, $16\frac{1}{2}$); (0, $16\frac{1}{2}$) and (1, 15); (3, $16\frac{1}{2}$) and (4, 15); (4, −1) and (4, $-2\frac{2}{5}$); (6, $-2\frac{2}{5}$) and (6, −4); ($-\frac{2}{5}$, 2) and ($-\frac{2}{5}$, 0).

Extra credit problems

18. Graph the following equations:

$$(x-2)^2+(y-3)^2=9/25;\ x^2+y^2-24y+144=4/25;\ (x+2)^2+(y-4)^2=16/25.$$

19. Graph all points (x, y) such that $\frac{2}{5}\geq x\geq -\frac{2}{5}$ and $2\geq y\geq 0$.

20. HAPPY NEW YEAR!

In the preceding test some of the exercises contain the same or similar instructions. It will save time and space to combine exercises such as Nos. 4, 7, and 17. However, these have been purposely separated in order that the final result of the test does not become evident too soon. Had these exercises not been so arranged the printed words or the picture might be given away. An attempt has been made to conceal the final result by having the lines and circles graphed in a disconnected sequence. It is of interest to note that some students manage to discover the secret of the test well in advance of completion while others will not notice the picture or words being formed until they are nearly finished. This might imply that some students are searching for "answers" that make sense and a solution to the entire problem.

For the teacher with a good imagination, there is no end to the possibilities for this method. Certainly some students with artistic leanings will find a great deal of enjoyment in a project developed around picture graphs. The author by no means intends to convey the impression that this is the way to teach graphing, but it does provide a way of "dressing up" some of the exercises.

The following set of exercises will serve to illustrate the application of this method to more advanced topics.

Graphing exercises

1. Graph $x^2=-2y$ for $-6\leq y\leq 0$.

2. Draw a line segment with slope zero from the point (−3, 5) that increases the abscissa by 3.4 units.

3 Graph

$$x^2+\left(y-\frac{21}{2}\right)^2=\left(\frac{3}{2}\right)^2.$$

4. Join the point (0.4, 5) and the point (1.8, 7) with a straight line segment.

5. Plot all points (x, y) such that $x=-4$, and $9\leq y\leq 12$.

6. Graph $x^2=-2y+2$ for $-6\leq y\leq 1$.

7. Join the point (−1.8, 7) and the point (−4, 9) with a straight line segment.

8. Graph

$$\frac{x^2}{4}+\frac{(y-5)^2}{16}=1.$$

9. Plot the following points:

$$(-3/5, 11),\ (3/5, 11),\ \left(0, \frac{21}{2}\right).$$

10. Draw a line segment one unit long that is parallel to the x-axis and is bisected by the point (0, 10).

• TESTING TIME

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

Using "take-home" tests

by Florence L. Elder, Junior-Senior High School, West Hempstead, Long Island

Editors' Note: The following is a contribution of which your editors feel doubly proud: first, because it is the first voluntary contribution to this department; secondly, because the ideas are novel as well as interesting. Mrs. Elder is chairman of the mathematics department at Junior-Senior High School in West Hempstead, Long Island. Last summer, Mrs. Elder was a study-group leader at the Fifth Mathematics Institute at Rutgers University.

A test should measure "the attainment of the objectives of instruction set down by the teacher."[1] The test items should be so constructed that the tester can determine the extent to which a student understands basic notions and principles. Yet how does the teacher measure a pupil's ability to apply understanding of principles, to generalize, to abstract? How does he measure the attainment of teaching objectives such as the following?

1. To recognize levels of ability
2. To challenge each member of the class
3. To encourage intellectual curiosity

Tests may be classified according to the type of test items included or with respect to the circumstances under which the test is taken. In addition to the usually employed classroom test there is the "take-home" or "open-book" type test.

"Take-home" tests

This brief proposes that "take-home" tests be used to measure the attainment of the objectives stated above. Before the test is constructed certain criteria should be established:

1. The test should not be "more of the same kind of work."
2. It should not be merely based on skills and manipulations.
3. The work should be something mathematically worthwhile.
4. The test should enable students to evaluate their own understanding and desire to do some difficult work.
5. Each member of the class should be challenged.
6. There should be items that encourage intellectual curiosity.

Since the "open-book" type test stimulates individual study and research, ample time should be provided to permit the kind of thinking required. Students may work alone or in groups, but should be responsible for all work appearing on their papers. If library books are used, a bibliography should be included. Directions to students might take this form:

Part I: This part constitutes the minimum test. It should be completed by each student.

Part II: These should prove to be more interesting and exciting. It is hoped that you will complete at least one problem in this part. There is great satisfaction in accomplishment.

Suggested test items

1. Is the reasoning below correct? Using graph paper, give geometric interpretation.

[1] Fouch, Robert S., and Kalin, Robert, "Testing Time," The Mathematics Teacher, January 1957, p. 81.

$$\begin{aligned}(9\tfrac{1}{2})^2 &= (9+\tfrac{1}{2})^2 = (9+\tfrac{1}{2})(9+\tfrac{1}{2})\\ &= 9(9+\tfrac{1}{2})+\tfrac{1}{2}(9+\tfrac{1}{2})\\ &= 9^2+9/2+9/2+\tfrac{1}{4}\\ &= 9^2+9+\tfrac{1}{4}\\ &= 9(9+1)+\tfrac{1}{4}\\ &= 9(10)+\tfrac{1}{4}\end{aligned}$$

This item may be used in the 8th or 9th grade. Students should understand the fundamental properties of rationals and identify the reasoning as an application of the basic laws of operation. The geometric sketch enables the student to associate the abstract notions with the more familiar concept of area.

2. a. Make a graph to show what time after 8:00 P.M. the minute hand and the hour hand of a clock will be together. Find the time correct to the nearest second.
   b. Let h represent the hour and n the number of minutes after the hour. Write an open sentence in h and n expressing the relationship between h and n at the time when the minute hand and the hour hand are together.
   c. Graph this relation.
   d. Use this graph to solve the initial given problem. Compare the answer obtained (d) with the above (a).

The second question may be used in the 9th grade. The student needs to understand that the intersection of two loci is the required solution. He is asked to express the relationship between the components of the ordered pair (h, n). He must recognize this as an equation in two variables. By a sequence of directions the student is guided in setting up various algorisms for solving a given problem.

To the curious student this problem suggests similar ones: At what time after 8:00 P.M. will the minute hand and the hour hand of a clock be opposite each other? at right angles to each other?

Students should be encouraged to explore these and other situations. A thorough analysis of this problem requires the ability to apply an understanding of basic principles.

3. a. Consider the number represented by the numeral 3142_{12} (base 12).
   b. Assume that any natural number may be represented by a polynomial function of the form

$$P(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$$

   with x a positive integer, n a non-negative integer, and each a_i a non-negative integer.
   Write 3142_{12} as a polynomial function, $P(x).(x=12)$
   c. Using synthetic division find the value of $P(12)$.
   d. Find $P(x'+2)$. Express as a polynomial in x' with coefficients less than x'.
   e. Find $P(x''-3)$. Express as a polynomial function in x'' with coefficients less than x''.
   f. Represent 3142_{12} as an equivalent numeral
      1. with base 10.
      2. with base 15.
      3. with base 7.
   g. Generalize your findings.

This question is a challenge. It permits each student to operate at his own level of abstraction. Most pupils will readily do parts (a) and (f). However, in items (d), (e), and (g) the tester is given an opportunity to measure the individual's power to do relational thinking. The question should not ask for a recall of manipulation previously demonstrated in the classroom. The new situation will provide a pupil-learning experience and teacher-measuring instrument.

4. In triangle ABC, angle $A=30°$ and side $AB=3$. Let β represent angle B and y represent side BC.
   a. Consider the ordered pair (β,y). Write an open sentence expressing the relationship, R, between β and y.
   b. Specify the domain and the range of the relation R.
   c. Graph R. Discuss.
   d. Graph R^i, the relation inverse to R.
   e. For relations R and R^i, explain why the relation is, or is not, a function.

Students enjoy this exploratory experience. The terminology of the question will vary with that used within the classroom. In the more traditional classroom situation the directions might read:

$(a-c)$. Graph y as a function of β.

$(d-e)$. Graph the inverse relation. Explain why this is, or is not, a function.

In the 10th or 11th grade a form of this question may be used to introduce the notion of "the ambiguous case."

5. Two points have an obstruction between them so that it is impossible to lay a straightedge between them. Devise one or more constructions for drawing two lines, one through each point, that will fall in one straight line.

This is but one of many questions that will challenge and intrigue each student. There are no clues or "guiding directions."

The tester has the opportunity to ascertain the student's ability to apply principles he understands. Depth of insight will be revealed in the ingenuity of the constructions and proofs submitted.

Summary

This discussion is not intended to imply that all tests should be of the "take-home" type. It does propose that, when well constructed and administered, "take-home" tests are an instrument for measuring the degree of understanding of the pupil who may not have instantaneous response, or who experiences undue fear during classroom-administered tests. "Take-home" tests also enable the teacher to gain an insight into the student's ability to apply himself to more extended abstract problem situations and his ability to discover mathematical principles. They provide a means of determining such personal factors as the "drive" to accomplish, and "measure the attainment of the objectives of instruction set down by the teacher."[3]

[3] *Loc. cit.*

What's new?

BOOKS

Miscellaneous

Barauch Spinoza—The Road to Inner Freedom, ed. and with an introduction by Dagobert D. Runes. New York, Philosophical Library, Inc., 1957. Cloth, 215 pp., $3.00.

Cross-Number Puzzles (teacher edition), Louis Grant Brandes. Portland, Maine, J. Weston Walch, 1957. Paper, iv+226 pp., $2.50.

Ernest Rutherford, Atom Pioneer, John Rowland. New York, Philosophical Library, Inc., 1957. Cloth, 160 pp., $4.75.

Galactic Nebulae and Interstellar Matter, Jean Dufay (trans. by A. J. Pomerans). New York, Philosophical Library, Inc., 1957. Cloth, 352 pp., $15.00.

How to Solve It (second edition), G. Polya. New York, Doubleday and Company, 1957. Paper, xxi+253 pp., $0.95.

Insight—A Study of Human Understanding. Bernard J. F. Lonergan. New York, Philosophical Library, Inc., 1956. Cloth, xxx+785 pp., $10.00.

The College Entrance Examination Board 55th Report of the Director. Princeton, New Jersey, College Entrance Examination Board, 1956 Paper, 133 pp., $0.50.

BOOKLETS

About to Choose Your High School Courses?, published by The President's Committee on Scientists and Engineers, Washington 25, D.C. For bulk orders, order from: William Wales, Darby Printing Co., 24th and Douglas Streets, N.E., Washington 18, D.C. Eight-page leaflet, $17.00 per thousand, minimum order of 500. Complimentary copies in small quantities.

Employment Opportunities for Women Mathematicians and Statisticians (Catalog No. L 13.3:363), Superintendent of Documents, U. S. Government Printing Office, Washington 25, D.C. 37 pp., 25 cents.

Improving Science and Mathematics Education, published by The President's Committee on Scientists and Engineers, Washington 25, D.C. For bulk orders, order from: William Wales, Darby Printing Co., 24th and Douglas Streets, N.E., Washington 18, D.C. 16 pp., 10 cents each, minimum order of 100; bulk orders, $85 per thousand. Complimentary copies in small quantities.

Report of the Membership Committee

Mary C. Rogers, Chairman, Membership Committee,
Roosevelt Junior High School, Westfield, New Jersey

The Membership Committee of the National Council of Teachers of Mathematics is happy to announce that an all-time high in membership total has been reached and wishes to thank you for your fine co-operation and support. It has been the individual assistance of each of you that has been largely responsible for this accomplishment. The Committee is indeed grateful to you for your helpfulness in this service to the Council.

We submit to you at this time a brief report of membership action taken at the Thirty-fifth Annual Meeting in Philadelphia and the latest membership analysis by states received from the Washington Office, based on the May 1, 1957 membership count.

Convention reactions

The announcement of a 15,175 membership total—based on the February 1, 1957 official count—was received with a great deal of enthusiasm by the Delegate Assembly, the State Representatives, and the many other persons who attended this, the largest annual meeting in the history of the Council.

Dr. Howard F. Fehr and the Board of Directors of the Council were particularly pleased with what they called the "phenomenal" growth over the past three years. They instructed the Committee to continue the work, using its own best judgment as to immediate goals and procedures. They expressed their sincere appreciation not only for the work of the Committee, but also for the outstanding support and diligent assistance of *all* persons concerned with increased membership. This includes not only the State Representatives and officers of Affiliated Groups, but also each of you whose direct contact with prospective members has been so instrumental in bringing about membership growth.

At the Delegate Assembly, Dr. Fehr expressed his thanks and that of the National Council Board to all persons responsible for the current strong and steady membership growth. He offered his opinion, however, that we had "only scratched the surface" of potential membership strength and challenged the group to an ultimate 25,000 membership total.

In her report to the Assembly, the Membership Chairman called for reactions to this challenge. Should present goals be maintained with continued efforts to increase membership? Should the 25,000 membership goal be accepted with State

goals raised accordingly? If so, what time limit should be agreed upon in which to reach new goals?

The Assembly voted to accept the 25,000 challenge at once and to support whatever plan for reaching this goal that might be recommended by the Membership Committee.

The Membership Committee, in conference with the State Representatives, arrived at a recommendation for a five-year time limit. This recommendation, approved by the NCTM Board, received generous publicity throughout the Convention. This action set up the following goals for the next five years:

April 1957–April 1958	17,000
April 1958–April 1959	19,000
April 1959–April 1960	21,000
April 1960–April 1961	23,000
April 1961–April 1962	25,000

RECORD OF MEMBERSHIP GROWTH

You will be most gratified to note the present high membership total is 16,181, based on the May 1, 1957 official count; also that this total is 95% of the suggested April 1, 1958 goal of 17,000. In preparing the accompanying membership analysis, we have reviewed former State goals, based on the 15,000 total. We have also indicated new State goals based on the 17,000 total to be reached in April 1958. We are reporting specific achievements based on both of these goals. A study of the data makes it quite evident that we are ready to "raise our sights" and that our new goals are very reasonable. With your continued support and assistance they should be readily reached.

Based on the 15,000 goal, 62% of all states and territories have reached their goals or gone beyond them. These states and territories are:

Arizona	275%
Oregon	172%
Nevada	164%
South Dakota	156%
Canada	148%
Utah	133%
California	132%
New Hampshire	126%
Florida	121%
U. S. Possessions	120%
Pennsylvania	119%
Washington	118%
Idaho	117%
New York	117%
Colorado	116%
Texas	116%
Connecticut	114%
Maine	113%
Montana	113%
Wisconsin	112%
Mississippi	111%
Foreign	109%
District of Columbia	107%
Michigan	107%
New Jersey	107%
Maryland	106%
Indiana	103%
Oklahoma	103%
Louisiana	101%
Ohio	101%
Iowa	100%
Massachusetts	100%

Based on this same 15,000 goal, membership achievement in these 12 states shows 90%–99% of established goals:

Georgia	99%
Virginia	99%
Delaware	97%
New Mexico	97%
Illinois	96%
Kansas	96%
Missouri	96%
Minnesota	94%
Kentucky	93%
Vermont	93%
Wyoming	92%
Arkansas	91%

Once again based on the 15,000 goal, membership in the following three states shows 85%–89% achievement:

Tennessee	87%
Alabama	86%
Nebraska	86%

Membership in five states has remained relatively low. Apparently in these states there is need for more stimulation of interest, more encouragement, and more help from the Membership Committee. Suggestions from present members in these states as to better procedures and more effective assistance from the Committee will be gratefully received.

States and Territories That Have Reached or Gone Beyond Their Goals

(Based on 1957–1958 Goals)

Arizona	243%	Idaho	105%
Nevada	153%	Pennsylvania	105%
Oregon	152%	Washington	105%
South Dakota	137%	New York	103%
Canada	130%	Colorado	102%
Utah	119%	Texas	102%
California	117%	Connecticut	100%
New Hampshire	112%	Maine	100%
Hawaii & U. S. Poss.	107%	Montana	100%
Florida	106%		

Membership Achievement 90%-99%

(Based on 1957–1958 Goals)

Mississippi	99%	New Jersey	94%
Wisconsin	99%	Maryland	93%
Foreign	96%	Oklahoma	92%
District of Columbia	95%	Indiana	91%
Michigan	94%		

Membership Achievement 85%-89%

(Based on 1957–1958 Goals)

Iowa	89%	Delaware	86%
Louisiana	89%	New Mexico	86%
Massachusetts	89%	Illinois	85%
Ohio	89%	Kansas	85%
Georgia	88%	Missouri	85%
Virginia	87%		

Leaders in Membership Totals Including Subscriptions

New York	1,378	Indiana	557
California	1,163	Wisconsin	470
Illinois	1,143	Massachusetts	442
Pennsylvania	1,093	Florida	423
Texas	778	Foreign	368
Ohio	747	Minnesota	368
Michigan	640	Virginia	352
New Jersey	598		

Leaders in Membership Totals—*Not* Including Subscriptions

Illinois	954	New Jersey	433
New York	947	Wisconsin	356
Pennsylvania	817	Florida	330
California	652	Massachusetts	330
Ohio	605	Virginia	272
Texas	527	Minnesota	254
Indiana	491	Kansas	248
Michigan	452		

National Council of Teachers of Mathematics
Analysis of Membership Growth—Members & Subscribers—May 1956-May 1957

	May 1956	May 1957 Individuals	May 1957 Totals	Goals '54–'57	%	Goals '57–'58	%
Alabama	135	107	147	170	86%	192	77%
Arizona	106	125	165	40	275%	68	243%
Arkansas	106	126	156	171	91%	194	80%
California	888	652	1,163	879	132%	996	117%
Colorado	168	159	208	180	116%	204	102%
Connecticut	220	184	259	228	114%	258	100%
Delaware	66	56	69	71	97%	80	86%
Dist. of Columbia	214	175	198	185	107%	209	95%
Florida	346	330	423	351	121%	398	106%
Georgia	152	133	185	186	99%	211	88%
Idaho	12	12	21	18	117%	20	105%
Illinois	1,075	954	1,143	1,188	96%	1,346	85%
Indiana	478	491	557	542	103%	614	91%
Iowa	270	215	297	296	100%	335	89%
Kansas	285	248	299	311	96%	352	85%
Kentucky	98	87	115	123	93%	139	83%
Louisiana	236	191	279	276	101%	313	89%
Maine	67	63	78	69	113%	78	100%
Maryland	269	245	301	285	106%	323	93%
Massachusetts	389	330	442	440	100%	498	89%
Michigan	544	452	640	600	107%	680	94%
Minnesota	311	254	368	391	94%	454	81%
Mississippi	106	102	137	123	111%	139	99%
Missouri	278	236	316	330	96%	374	85%
Montana	59	48	68	60	113%	68	100%
Nebraska	126	109	142	165	86%	187	76%
Nevada	23	14	23	14	164%	15	153%
New Hampshire	74	64	82	65	126%	73	112%
New Jersey	511	433	598	561	107%	636	94%
New Mexico	79	58	86	89	97%	100	86%
New York	1,152	947	1,378	1,181	117%	1,338	103%
North Carolina	198	161	209	255	82%	289	72%
North Dakota	28	22	35	44	80%	49	72%
Ohio	613	605	747	740	101%	838	89%
Oklahoma	212	183	245	237	103%	265	92%
Oregon	176	151	230	134	172%	151	152%
Pennsylvania	844	817	1,093	920	119%	1,042	105%
Rhode Island	52	34	55	71	77%	80	69%
South Carolina	102	51	105	137	77%	155	68%
South Dakota	52	31	56	36	156%	41	137%
Tennessee	207	161	213	246	87%	279	76%
Texas	673	527	778	672	116%	762	102%
Utah	60	39	64	48	133%	54	119%
Vermont	41	31	38	41	93%	46	83%
Virginia	310	272	352	357	99%	405	87%
Washington	226	161	270	228	118%	258	105%
West Virginia	77	73	91	158	58%	179	51%
Wisconsin	465	356	470	420	112%	476	99%
Wyoming	36	26	33	36	92%	41	80%
TOTALS	13,214	11,301	15,427	14,388	107%	16,305	93%
Hawaii & U. S. Poss.	139	60	89	74	120%	83	107%
Canada	233	175	297	201	148%	228	130%
Foreign	241	145	368	339	109%	384	96%
GRAND TOTALS	13,827	11,681	16,181	15,002	108%	17,000	95%

Leaders in Membership Growth—Including Subscriptions

(Since May 1956)

California	275	Indiana	79
Pennsylvania	249	Florida	77
New York	226	Illinois	68
Ohio	134	Canada	64
Foreign	127	Arizona	59
Texas	105	Minnesota	57
Michigan	96	Oregon	54
New Jersey	87		

Greatest Relative Growth

(Since May 1956)

Idaho	Oregon	Colorado
Arizona	Pennsylvania	Florida
Foreign	Mississippi	Georgia
Arkansas	Canada	Ohio
California	North Dakota	Washington

States with Continuous Growth

(Since May 1956)

Arkansas	Kentucky	Oklahoma
California	Mississippi	Oregon
Connecticut	Nebraska	Pennsylvania
Illinois	Ohio	Utah

FUTURE PLANS

We believe this record of accomplishment is one of which each of us should be justifiably proud. We are confident that the future holds achievements equally commendable; that with your continued support and assistance, even the most ambitious goals will be reached.

We suggest that future procedures should continue to follow closely those that have proved most effective in the past.

1. We have found the "Each One Win One" technique to be our most valuable aid. We urge a continuance of its use by all present members of the Council.
2. We cannot emphasize too strongly the importance of the National Council publications to the mathematics teacher. Currently invaluable is the Twenty-third Yearbook—*Insights into Modern Mathematics*. Special prices are given to NCTM members on all Council yearbooks.
3. Your prompt renewals greatly facilitate the keeping of records in the Washington Office and the preparation of reports.
4. You, who are members of the mathematics staff in colleges of education and similar education centers, can be, and are, of invaluable assistance through your continued stimulation of interest in NCTM services among your students.
5. Similarly, you supervisors and/or mathematics department chairmen are obtaining increasingly fine results in your work with your teachers. Keep up the good work.
6. Many state and other local associations of mathematics teachers, together with the state representatives to the National Council, are performing an outstanding service to the Council. A continuance and expansion of your fine work will be most sincerely appreciated.
7. Attention should be called to the *National Council Membership Direc-*

tory and the professional prestige afforded through inclusion in this listing.

8. Library and other institutional subscriptions will still be included in preparing reports of membership totals.

The personnel of the Membership Committee remains much the same as during the past three years. The people presently serving in this capacity are:

Nellie Alexander
Illinois, Indiana, Iowa, Ohio, West Virginia
Pearl Bond
Alabama, Louisiana, Mississippi, Texas
Marian C. Cliffe
Arizona, California, New Mexico, Utah
Janet Height
New England States
Mary Lee Foster
Arkansas, Kentucky, Missouri, Nevada, Tennessee
Mary Reed
Michigan, Minnesota, Ontario, South Dakota, Wisconsin
Harold J. Hunt
Idaho, Montana, North Dakota, Oregon, Washington
Florence Ingham
Colorado, Kansas, Nebraska, Oklahoma, Wyoming
Faith Novinger
Delaware, District of Columbia, New Jersey, New York, Pennsylvania
Bess Patton
Florida, Georgia, Maryland, North Carolina, South Carolina, Virginia
Myrl H. Ahrendt and Elizabeth Roudebush, ex officio members
Mary C. Rogers, Chairman

Each of these persons is performing an outstanding service for the National Council in its work with State Representatives and with Presidents of the Affiliated Groups in their respective territories.

Please accept our best wishes for your greatest professional success.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING
April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Human, Board of Education, Cleveland, Ohio

JOINT MEETING WITH NEA
June 30, 1958
Cleveland, Ohio
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

EIGHTEENTH SUMMER MEETING
August 19–20, 1958
Colorado State College of Education, Greeley, Colorado
Forest N. Fisch, Colorado State College of Education, Greeley, Colorado

• NOTES FROM THE WASHINGTON OFFICE

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Below are reports of registrations at the Seventeenth Christmas Meeting and the Thirty-fifth Annual Meeting. The Seventeenth Christmas Meeting was the first meeting of the Council ever held in the state of Arkansas. The hospitality and spirit of co-operation shown by our Arkansas members will long be remembered by those who attended the convention. The Thirty-fifth Annual Meeting was the *largest* ever held by the Council, with 1,464 official registrations. The largest previous meeting was in Chicago in April 1950, when 1,322 persons registered.

The official registration report contains only those who registered for the purpose of attending the convention and, therefore, does not include members and friends of families or persons who were present for other reasons.

Registrations at the Seventeenth Christmas Meeting

National Council of Teachers of Mathematics, Jonesboro, Arkansas, December 27–29, 1956

Arkansas	80	Nebraska	6
California	3	New Hampshire	1
Colorado	3	New Jersey	5
District of Columbia	3	New Mexico	1
Florida	5	New York	2
Illinois	23	North Carolina	1
Indiana	7	Ohio	1
Iowa	8	Oklahoma	5
Kansas	13	Pennsylvania	1
Kentucky	2	South Carolina	2
Louisiana	10	South Dakota	1
Maryland	1	Tennessee	22
Massachusetts	1	Texas	25
Michigan	5	Virginia	1
Minnesota	4	Wisconsin	10
Mississippi	5		
Missouri	24	Total	283
Montana	2		

Registrations at the Thirty-fifth Annual Meeting

National Council of Teachers of Mathematics, Philadelphia, Pennsylvania, March 28–30, 1957

Alabama	3	Maine	2
Arkansas	2	Maryland	118
California	13	Massachusetts	37
Connecticut	27	Michigan	23
Delaware	24	Minnesota	6
District of Columbia	56	Mississippi	2
Florida	13	Missouri	6
Georgia	7	Nebraska	5
Illinois	46	New Hampshire	8
Indiana	32	New Jersey	229
Iowa	7	New York	151
Kansas	3	North Carolina	7
Kentucky	2	Ohio	61
Louisiana	5	Oklahoma	2

Oregon	2	Washington	2
Pennsylvania	464	West Virginia	8
Rhode Island	4	Wisconsin	12
South Carolina	1	Puerto Rico	1
Tennessee	6	Canada	17
Texas	8		
Vermont	2	Total	1,464
Virginia	40		

Committees of the National Council of Teachers of Mathematics (1957-1958)

National Council Representatives

Executive Secretary

Myrl H. Ahrendt, Washington, D. C.

AAAS Cooperative Committee on Science and Mathematics

Henry W. Syer, Boston University, Boston, Mass. (1958)

Education Advisory Committee to Science Service

Veryl Schult, Washington, D. C. (1958)

Policy Committee of Mathematical Associations

Howard F. Fehr, Teachers College, Columbia U., New York (1959)

John R. Mayor, Washington, D. C.

United States Commission on Mathematics Instruction

Howard F. Fehr, Teachers College, Columbia U., New York (1960)

Henry W. Syer, Boston University, Boston, Mass. (1958)

Secretary of the Board

Houston T. Karnes, Baton Rouge, La. (1958)

Affiliated Groups

Elizabeth Roudebush, Seattle, Washington, Chairman (1959)

Northeastern: Catherine Lyons, Pittsburgh, Pa. (1958)

North Central: Virigina Pratt, Omaha, Neb. (1959)

Western: Lesta Hoel, Portland, Ore. (1959)

Southeastern: Houston Banks, Nashville, Tenn. (1958)

Central: Adeline A. Riefling, St. Louis, Mo. (1959)

Southwestern: Eunice Lewis, Norman, Okla. (1959)

Auditing

Ethel H. Grubbs, Washington, D. C. (1958)

* Julia E. Diggins, Washington, D. C. (1959)

Duties: To make general audit of the accounts of the National Office.

* Signifies new appointee.

Budget

Houston Karnes, Baton Rouge, La., Chairman (1958)

Agnes Herbert, Baltimore, Md. (1959)

* Jackson Adkins, Exeter, N. H. (1960)

Duties: To draw up and present to the Board at its annual meeting the budget of the Council for the year 1958–1959.

Representatives to AAAS Executive Committee

Veryl Schult, Washington, D. C.

Phillip Jones, Ann Arbor, Mich.

Executive (1957–1958)

Howard F. Fehr, Teachers College, Columbia U., N. Y.

Jackson B. Adkins, Exeter, N. H.

Phillip Jones, Ann Arbor, Mich.

Membership (1958)

Mary Rogers, Westfield, N. J., Chairman

Myrl Ahrendt, Washington, D. C.

Elizabeth Roudebush, Seattle, Wash.

* Marian C. Cliff, Glendale, Calif.

* Mary Rickey, Cedar Rapids, Iowa

Additional members to be selected by the five above.

Duties: To work to increase membership, working through the Affiliated Groups and State representatives. To suggest ways and means of increasing membership.

Relations with the NEA

Jane M. Hill, Washington, D. C., Chairman (1959)

Annie John Williams, Durham, N. C. (1958)

Ida May Bernhard, Austin, Tex. (1958)

William Gager, Gainesville, Fla. (1958)

Duties: To report on the interrelation and responsibility of services of NCTM to NEA and vice versa. To suggest policy on these interrelations.

Convention Committee
Glenn Ayre, Macomb, Ill., Chairman (1959)
Marguerite Brydegaard, San Diego, Calif. (1959)
Forest N. Fisch, Greeley, Colo. (1960)
Alice Hach, Racine, Wis. (1960)
* James Nedelman, Cupertino, Calif. (1959)

Duties: To study, plan and report on geographic and strategic locations of annual, summer, and other conventions. To recommend to the Board a planned sequence of convention cities projected to 1965.

Cleveland Convention
Lawrence Hyman, Cleveland, Ohio, Chairman (1959)
James D. Bristol, Cleveland, Ohio
Evelyn M. Coates, Cleveland, Ohio
James A. Gates, Cleveland, Ohio
Doris M. Drueger, Cleveland, Ohio
Harold S. Miller, Cleveland, Ohio
Alex Rubins, Cleveland, Ohio
Irwin N. Sokol, Cleveland, Ohio

Duties: To plan, organize and carry out all details except program for the annual convention, April 9–12, 1958.

Supplementary Publications
Henry Swain, Winnetka, Ill., Chairman (1958)
* Mildred Keiffer, Cincinnati, Ohio (1960)
Dwain Small, Carbondale, Ill. (1958)
Margaret Joseph, Milwaukee, Wis. (1959)
Edwin Eagle, San Diego, Calif. (1960)
* J. Houston Banks, Nashville, Tenn. (1960)
Marguerite Brydegaard, San Diego, Calif. (1958)
Kenneth Kidd, Gainesville, Fla. (1958)
Jesse Osborne, St. Louis, Mo. (1958)
Lawrence A. Ringenberg, Charleston, Ill. (1959)
Robert Seber, Kalamazoo, Mich. (1959)
* James Ulrich, Arlington Heights, Ill. (1960)

Duties: To solicit, receive, evaluate, and submit for publication, small publications other than the Journals and Yearbooks.

Yearbook Planning
Myron F. Rosskopf, Teachers College, Columbia U., N. Y., Chairman (1959)
Robert E. Pingry, Urbana, Ill. (1960)
* Bruce E. Meserve, Montclair, N. J. (1961)

Duties: To report on status of yearbooks under way, recommend further areas for yearbooks, and recommend possible editors and committee members for future yearbooks.

Publications Board
Robert Fouch, Tallahassee, Fla., Chairman (1958)
Glenn Ayre, Macomb, Ill., Chairman (1958–59)
* Clifford Bell, Los Angeles, Calif., Chairman (1959–60)
The Mathematics Student Journal, Max Beberman (1958)
THE MATHEMATICS TEACHER, Henry Van Engen (1959)
The Arithmetic Teacher, Ben Sueltz (1960)
Supplementary Publications, Henry Swain (1958)
Yearbook Planning, Myron Rosskopf (1959)

Duties: Defined in minutes of Board meeting, April 1956.

Mathematics for the Talented (1958)
Daniel B. Lloyd, Washington, D. C., Chairman
Mary Lee Foster, Arkadelphia, Ark.
Frances Johnson, Oneonta, N. Y.
Robert Vollmer, Battle Creek, Mich.
Glen Vanatta, Indianapolis, Ind.
Robert S. Fouch, Tallahassee, Fla.
Ruth Greenwald, Highland Park, Ill.

Duties: To study and propose policy for the NCTM in fostering the mathematics education of talented students.

Cooperation with Industry (1958)
Marie Wilcox, Indianapolis, Ind., Chairman
Kenneth Brown, Washington, D. C.
Phillip Jones, Ann Arbor, Mich.
Zeke Loflin, Lafayette, La.
William Glenn, Pasadena, Calif.
Lauren Woodby, Mt. Pleasant, Mich.
* Paul Gore, Gary, Ind.

Duties: To continue work as reported to the Board at Milwaukee 1956. To advise the Board on the duties, services, and advisability of continuing this committee.

Reporting Elections (1958)
* Clifford Bell, Los Angeles, Calif., Chairman
Myrl Ahrendt, Washington, D. C.
* Howard F. Fehr, New York, N. Y.

Duties: To certify results of election of the Board of Directors. To report results of election at annual business meeting.

International Relations
Veryl Schult, Washington, D. C., Chairman (1960)
E. H. C. Hildebrandt, Evanston, Ill. (1959)
Robert Rourke, Kent, Conn. (1959)
Alfred Putnam, Chicago, Ill. (1958)

Duties: To be given by the chairman.

Research
John Kinsella, New York, N. Y., Chairman (1959)
Kenneth Brown, Washington, D. C. (1958)
Nathan Lazar, Columbus, Ohio (1958)
* Clark Lay, Los Angeles, Calif. (1959)

Duties: To provide Research Section at annual convention, to prepare and to provide means of collecting and publishing research and to promote study of research in mathematics education.

Nomination of Editor for The Mathematics Student Journal (1958–61)
Max Beberman, Urbana, Ill., Chairman
Julius Hlavaty, New York, N. Y.
Kenneth Henderson, Urbana, Ill.

Duties: To nominate two persons who will accept an appointment by the Board as editor of *The Mathematics Student Journal.*

Nominations and Elections (1958)
Clifford Bell, Los Angeles, Calif., Chairman
Lynwood Wren, Nashville, Tenn.
Irene Sauble, Detroit, Mich.
William Gager, Gainesville, Fla.
Jackson Adkins, Exeter, N. H.
Mary Foster, Arkadelphia, Ark.
Veryl Schult, Washington, D. C.
Marie Wilcox, Indianapolis, Ind.

Duties: To secure and recommend nominations for spring, 1958, election.

Nominations and Elections (1959)
Milton Beckmann, Lincoln, Neb., Chairman
Clifford Bell, Los Angeles, Calif.
Charles Butler, Kalamazoo, Mich.
Robert Fouch, Tallahassee, Fla.
Martha Hildebrandt, Maywood, Ill.
Mildred Keiffer, Cincinnati, Ohio
Ann Peters, Keene, N. H.
Myron Rosskopf, New York, N. Y.
Marie Wilcox, Indianapolis, Ind.

Elementary School Curriculum
J. Fred Weaver, Boston, Mass., Chairman (1958)
Anne Peters, Keene, N. H. (1959)
Henry Van Engen, Cedar Falls, Iowa (1959)
Irene Sauble, Detroit, Mich. (1958)
* Laura Eads, New York, N. Y. (1960)
Joyce Benbrook, Houston, Tex. (1958)

Duties: To develop a program of study of the elementary school curriculum in mathematics, and to propose methods for initiating and supporting the study and its ultimate report.

Secondary School Curriculum
Frank B. Allen, La Grange, Ill., Chairman
Jackson B. Adkins, Exeter, N. H.
Howard F. Fehr, New York, N. Y.
A. S. Householder, Oak Ridge, Tenn.
Lottchen Hunter, Wichita, Kan.
Burton W. Jones, Boulder, Colo.
John R. Mayor, Washington, D. C.
Bruce Meserve, Montclair, N. J.
Sheldon Myers, Princeton, N. J.
E. B. Newell, Indianapolis, Ind.
Alfred Putnam, Chicago, Ill.
Elizabeth Roudebush, Seattle, Wash.
Helen M. Walker, New York, N. Y.
Marie Wilcox, Indianapolis, Ind.
Lynwood Wren, Nashville, Tenn.
Magnus Hestenes, Los Angeles, Calif. (Advisor)

Duties: To carry out proposed research on secondary school mathematics curriculum.

Yearbooks
Twenty-fourth (Mathematical Concepts)
Phillip Jones, Ann Arbor, Mich., Chairman
Harold Fawcett, Columbus, Ohio
Alice Hach, Racine, Wis.
Charlotte Junge, Detroit, Mich.
Henry Syer, Boston, Mass.
Henry Van Engen, Cedar Falls, Iowa
Twenty-fifth (Arithmetic)
Foster Grossnickle, Jersey City, N. J., Chairman
Dan Dawson, Stanford, Calif.
Ida Mae Heard, Lafayette, La.
Irene Sauble, Detroit, Mich.
Herbert Spitzer, Ames, Iowa
Louis C. Thiele, Detroit, Mich.
Twenty-sixth (Evaluation)
Donovan Johnson, Minneapolis, Minn., Chairman

Television
Joe Hooten, Tallahassee, Fla., Chairman (1960)
Lewis Scholl, Buffalo, N. Y. (1959)
Sylvia Vopni, Seattle, Wash. (1959)
Phillip Jones, Ann Arbor, Mich. (1958)
David Wells, Omaha, Neb. (1960)
George Anderson, Millersville, Pa., 1960

Duties: To investigate and report on the status of television in the teaching of mathematics. To propose action and the stand that the NCTM should take on the use of TV in mathematics education.

Teacher Education, Certification and Recruitment
Kenneth Brown, Washington, D. C., Chairman (1960)
Eugene Northrop, Chicago, Ill. (1958)
Myron Rosskopf, New York, N. Y. (1958)
Robert Kalin, Tallahassee, Fla. (1959)
David Page, Urbana, Ill. (1960)
Charles Atherton, Shepherdstown, W. Va. (1959)
Richard Purdy, San Jose, Calif. (1959)

Duties: To study the present certification requirements of the States of the Union and to recommend action and future study to the Board.

The Mathematics Teacher (1958)
Henry Van Engen Cedar Falls, Iowa, Editor (1959)
Irvin Brune, Cedar Falls, Iowa, Assistant Editor
Jackson B. Adkins, Exeter, N. H.
Mildred Keiffer, Cincinnati, Ohio
Z. L. Loflin, Lafayette, La.
Philip Peak, Bloomington, Ind.
Ernest Ranucci, Newark, N. J.
Myron F. Rosskopf, New York, N. Y.

The Mathematics Student Journal (1958)
Max Beberman, Urbana, Ill., Editor
L. J. Adams, Santa Monica, Calif.
Izark Wirszup, Chicago, Ill.

The Arithmetic Teacher (1958)
Ben A. Sueltz, Cortland, N. Y., Editor (1960)
John R. Clark, New Hope, Pa.
Marguerite Brydegaard, San Diego, Calif.

National Association of Secondary School Principals Bulletin (1959)

Myron F. Rosskopf, New York, N. Y., Chairman
Eugene Smith, Columbus, Ohio
Robert Fouch, Tallahassee, Fla.
Glenn Ayre, Macomb, Ill.
Clark Lay, Los Angeles, Calif.
Ida May Bernhard, Austin, Tex.

Duties: To secure writers and edit March 1959 BULLETIN of NASSP

Committee on Information Center

John R. Mayor, Washington, D. C., Chairman (AAAS)
Myrl Ahrendt, Washington, D. C. (NCTM)
Howard F. Fehr, New York, N. Y. (ex officio)
G. Bailey Price, Lawrence, Kan. (MAA)
Albert E. Meder, New Brunswick, N. J. (AMS)

Duties: To investigate the purposes, means, and structure of an information gathering and disseminating agency on mathematics education.

Mathematical profiles

Continued from page 512

that a good contribution toward making mathematics interesting might be made by attempting to make mathematicians interesting. It seemed that this could be done best by presenting sketches, or profiles in the New Yorker sense, of real, live mathematicians.

Since the profiles are to be designed for high school students, it seemed clear that the mathematicians profiled should not be too old, should be interesting and engaged in interesting work, and should be outstanding. Further, there should be women as well as men. Since many high school students are apt to consider a doctor's degree quite unattainable, many if not most of the mathematicians should be at the bachelor's or master's level academically. And since the object is to make it known that nonacademic opportunities exist, most should be in nonacademic employment.

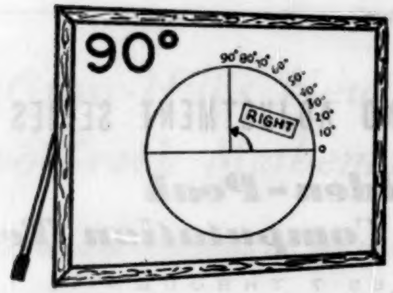
It is hoped that eventually these profiles will be assembled and printed in a little brochure, to be made available at no cost to high school students, as well as their teachers and counselors. With this in mind, funds were requested from the National Science Foundation to finance the necessary interviews, writing, printing, and distribution. At present, funds have been made available for interviewing and writing, and it is to be hoped that the funds for distribution will be forthcoming when needed.

As can be imagined, the task of selecting the profilees was not a simple one, and the Committee does not claim to have made the best possible choices. It was thought that eight or ten profiles would be sufficiently representative, and that a larger number might repel the prospective reader. But to ameliorate the difficulties somewhat, twenty names were chosen, and the profiles to be published in the brochure will be selected from among the twenty that will actually be prepared. For selecting these twenty, letters were sent to a number of governmental and industrial employers of mathematicians, requesting nominations with a brief sketch of the background, interests, and achievements of each nominee. Responses to these letters, together with some nominations by members of the Committee, yielded a list of about fifty or so, and the Committee voted on these.

As plans were being formulated, it came to the attention of the Committee that the National Council of Teachers of Mathematics had requested funds from the National Science Foundation for preparing and distributing a brochure setting forth the cultural values of mathematics and the mathematical requirements of various occupations. After some discussion between the two groups, it was agreed to join forces and to include the statement of requirements in the same brochure with the profiles.

The interviewing and writing are now under way, and are being done by a professional writer engaged for the purpose by the National Research Council. While funds for printing and distribution are not yet assured, it is to be hoped that they can be had, or that some other means of publication can be found. The present article is written to inform others interested in these problems of one specific endeavor that is being made to improve the future level of applied mathematics. Moreover, the present chairman of the Committee on Applications of Mathematics will welcome any suggestions for future projects, and doubtless the same will hold for his successors.

A. S. Householder, Chairman
Committee on Mathematics
National Research Council
Oak Ridge, Tennessee



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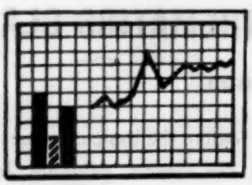

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